§ Two Random Variables

If we can define one RV on \((\mathcal{S}, \mathcal{F}, \mathbb{P})\), why not extend the idea and define two RVs.

\begin{equation}
\begin{aligned}
\mathcal{S} &\xrightarrow{\mathcal{F}} \mathcal{Z} \xrightarrow{\mathbb{P}} \mathcal{X} \\
\mathcal{Z}(w_1) &\xrightarrow{\mathcal{F}} \mathcal{Z}(w_2) \xrightarrow{\mathcal{F}} \mathcal{Z}(w_k) \\
\mathcal{P}(w_1) &\xrightarrow{\mathcal{F}} \mathcal{P}(w_2) \xrightarrow{\mathcal{F}} \mathcal{P}(w_k)
\end{aligned}
\end{equation}

i.e.

\begin{equation}
\begin{aligned}
(\mathcal{S}, \mathcal{F}, \mathbb{P}) &\xrightarrow{\mathcal{X}} (A, B(A), \mathbb{P}_A) \\
&\xrightarrow{\mathcal{Y}} (B, B(B), \mathbb{P}_B)
\end{aligned}
\end{equation}

We can also think of the pair of RVs as a mapping from \(\mathcal{S}\) to \(\mathbb{R}^2\)

\begin{equation}
\begin{aligned}
(\mathcal{S}(\cdot), \mathcal{P}(\cdot)) : \mathcal{S} &\longrightarrow \mathbb{R}^2
\end{aligned}
\end{equation}
Although the cdf $F_X(x)$ and $F_Y(y)$ describe prob. behavior of $X$ and $Y$ separately, they do **not** describe occurrence of $X$ and $Y$ together ($P_{XY}$)

Consider a set $D \subseteq \mathbb{R}^2$, we will assume $D$ can be written as countable union of open rectangles and their complements in the plane.

$$D \subseteq B(\mathbb{R}^2) = \text{Smallest } \sigma\text{-algebra containing open rectangles.}$$

We would like to compute the prob. of the event

$$\{ (X, Y) \in D \} = \{ W: (X(W), Y(W)) \in D \}$$

Knowing $F_X(x)$ and $F_Y(y)$ is not enough to do this, we need a "joint cdf"

**Joint CDF**

**Defn:** The joint cdf of two RVs $X$ and $Y$ defined on $(\mathcal{S}, \mathcal{F}, P)$ is the prob. of the event

$$\{ X \leq x \} \cap \{ Y \leq y \}, \ x, y \in \mathbb{R}$$

i.e. $F_{XY}(x,y) = P(\{ X \leq x \} \cap \{ Y \leq y \})$
n.b. We can think of \( F_{\mathbb{P}}(x,y) \) as
\[
F_{\mathbb{P}}(x,y) = P\left\{ (\alpha, \beta) \in D_1(x,y) \right\}
\]

![Diagram of a shaded region in the x-y plane, with the domain \( D_1(x,y) \) highlighted.]

\[ D_1(x,y) = \{ (\alpha, \beta) : \alpha \leq x \quad \text{and} \quad \beta \leq y \} \]

We will shorten notation.
\[
\{ \alpha \leq x, \beta \leq y \} = \{ \alpha \leq y \} \cap \{ \beta \leq y \}
\]
\[
\Rightarrow F_{\mathbb{P}}(x,y) = P\left( \{ \alpha \leq x, \beta \leq y \} \right)
\]

Properties of the Joint CDF:

1. \( F_{\mathbb{P}}(-\infty, y) = 0 \), \( F_{\mathbb{P}}(x, -\infty) = 0 \)
   \[
   F_{\mathbb{P}}(\infty, y) = F_y(y), \quad F_{\mathbb{P}}(x, \infty) = F_x(x)
   \]
   \[
   F_{\mathbb{P}}(\infty, \infty) = 1.
   \]

2. \( P(\{ \alpha < \alpha \leq \alpha_2, \beta \leq \beta_2 \}) = F_{\mathbb{P}}(x_2,y) - F_{\mathbb{P}}(x_1,y) \)
   \[
   P(\{ \alpha \leq x, \beta_1 \leq \beta \leq \beta_2 \}) = F_{\mathbb{P}}(x,y_2) - F_{\mathbb{P}}(x,y_1)
   \]
3. \( P(\{ x_1 \leq X \leq x_2, \, y_1 \leq Y \leq y_2 \}) \)
   
   \[
   = \bar{F}_X(x_2, y_2) - \bar{F}_X(x_1, y_2) - \bar{F}_X(x_2, y_1) + \bar{F}_X(x_1, y_1)
   \]

**Joint PDF**

**Defn:** The joint PDF of two RVs \( X \) and \( Y \) on \((\mathcal{S}, \mathcal{F}, P)\) is \( f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y} \).

From the properties of the joint CDF and the derivative, it follows that:

\[
\begin{cases}
(i) \quad f_{XY}(x, y) \geq 0, \quad \forall x, y \in \mathbb{R} \\
(ii) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) \, dx \, dy = 1. \\
(iii) \quad F_{XY}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(x', y') \, dx' \, dy'
\end{cases}
\]
In general, for $D \subseteq \mathbb{R}^2$ ($D \in \mathcal{B}((\mathbb{R}^2))$), we have
\[
P\left(\{(x,y) \in D\}\right) = \iint_D f_{Z,Y}(x,y) \, dx \, dy
\]
\[
= \iint_{\mathbb{R}^2} f_{Z,Y}(x,y) \cdot 1_D((x,y)) \, dx \, dy
\]

Again, recall
\[
\{(x,y) \in D\} = \{w: (X(w), Y(w)) \in D\} \subset \mathcal{F}
\]
\[
\in \mathcal{F}
\]

Two RVs $X$ and $Y$ defined on the same $(\mathcal{F}, \mathbb{P})$ are called \underline{jointly distributed}. They have joint cdf $F_{Z,Y}(x,y)$ and joint pdf $f_{Z,Y}(x,y)$.

\underline{Marginal Probabilities}:

- When studying two (or more) RVs, the cdfs, pdfs, and "statistics" of a single RV are referred to as "marginal."

- Given the joint cdf $F_{Z,Y}(x,y)$:
  \[
  F_X(x) = F_{Z,Y}(x, \infty)
  \]
  \[
  F_Y(y) = F_{Z,Y}(\infty, y)
  \]
If \( f_{X,Y}(x,y) \) is joint pdf of \( X \) and \( Y \),

\[
    f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dy
\]

\[
    f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx
\]

\text{n.b.} Given \( f_{X,Y}(x,y) \), we can find marginals, but given \( f_X(x) \), \( f_Y(y) \), we cannot find \( f_{X,Y}(x,y) \)

**Defn:** Two jointly distributed RVs \( X \) and \( Y \) are \text{jointly Gaussian} if the joint pdf has the form

\[
    f_{X,Y}(x,y) = \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1-r^2}} \exp \left\{ \frac{-1}{2(1-r^2)} \left[ \frac{(x-\mu_x)^2}{\sigma_x^2} \right. \left. - 2r \frac{(x-\mu_x)(y-\mu_y)}{\sigma_x \sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2} \right] \right\}
\]

where \( \mu_x, \mu_y \in \mathbb{R} \), \( \sigma_x, \sigma_y > 0 \), and \(-1 \leq r \leq 1\)

\text{n.b.} Two jointly distributed Gaussian RVs \( X \) and \( Y \) (with above \( f_{X,Y}(x,y) \)) are marginally Gaussian:

\[
    f_X(x) = \frac{1}{\sqrt{2\pi} \sigma_x} \exp \left\{ \frac{-(x-\mu_x)^2}{2\sigma_x^2} \right\}
\]

\[
    f_Y(y) = \frac{1}{\sqrt{2\pi} \sigma_y} \exp \left\{ \frac{-(y-\mu_y)^2}{2\sigma_y^2} \right\}
\]

\text{The Converse may not be true.}
Statistically Independence

Defn: Two RVs $X$ and $Y$ on $(\mathcal{F}, \mathbb{P})$ are **statistically independent** if the events $A = (-\infty, x]$ and $B = (-\infty, y]$ are statistically independent events for all $A, B \in \mathcal{B}(\mathbb{R})$.

If we take $A = \{x \leq X\}$ and $B = \{y \leq Y\}$, $X, Y \in \mathbb{R}$.
We note that if $X$ and $Y$ are stat. indep.

$$F_{XY}(x, y) = P(\{X \leq x\} \cap \{Y \leq y\}) = P(\{X \leq x\}) \cdot P(\{Y \leq y\}) = F_X(x) F_Y(y)$$

and

$$f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y} = \frac{\partial F_X(x)}{\partial x} \cdot \frac{\partial F_Y(y)}{\partial y} = f_X(x) f_Y(y)$$

It can also be shown that if $f_{XY}(x, y) = f_X(x) f_Y(y)$, then $X$ and $Y$ are stat. indep.

Proof

For $A, B \in \mathcal{B}(\mathbb{R})$

$$P(\{x \in A\} \cap \{y \in B\}) = \int \int f_{XY}(x, y) \, dx \, dy$$

$$= \int \int f_X(x) f_Y(y) \, dx \, dy = \int f_X(x) \, dx \int f_Y(y) \, dy$$

$$= P(\{x \in A\}) \cdot P(\{y \in B\})$$
Thus, we could take as an alternative defn:

**Defn**: Two jointly distributed RVs, \( X \) and \( Y \) are stat. indep. iff \( f_{XY}(x,y) = f_X(x) f_Y(y) \)

**Lemma**: \( X \) and \( Y \) are stat. indep. iff there exist functions \( g(x), h(y) \) such that \( f_{XY}(x,y) = g(x) h(y) \)

**Defn**: Let \( X \) and \( Y \) be two RVs on \((S, F, P)\). Let \( A \in F \). Then \( X \) and \( Y \) are conditionally independent conditioned on \( A \) iff \( f_{XY}(x,y|A) = f_X(x|A) f_Y(y|A) \)

**N.B.** Conditionally indep. of \( X \) and \( Y \) conditioned on \( A \in F \) does not in general imply stat. indep. of \( X \) and \( Y \)

**Theorem**: If two jointly dist. RV, \( X \) and \( Y \) are stat. indep., then the RVs \( Z = g(X) \) and \( W = h(Y) \) are stat. indep.

Suppose we have RV, \( X \) on \((S_1, F_1, P_1)\) and RV \( Y \) on \((S_2, F_2, P_2)\)

We can form the combined experiment \((S, F, P)\), \( S = S_1 \times S_2 \)

\( F = \sigma (\{ A \times B : A \in F_1, B \in F_2 \}) \). \( P \): An appropriate prob. measure. Consistent w/ \( P_1 \) and \( P_2 \). Then \( X \) and \( Y \) can both be viewed as mapping from \((S, F, P)\) to \( R \).

**Theorem**: If random experiment \((S_1, F_1, P_1)\) and \((S_2, F_2, P_2)\) are independent, the jointly dist. RVs \( X \) and \( Y \) on \((S, F, P)\) are stat. indep.
Conditional Distribution

Conditional Distributions:

**Defn:** Let \( X \) and \( Y \) be two RVs on \((\mathcal{S}, F, P)\).

The joint conditional cdf of \( X \) and \( Y \) conditioned on \( M \in F \), is

\[
F_{X|Y}(x|y|M) = \frac{P(\{X \leq x\} \cap \{Y \leq y\} \cap M)}{P(M)}
\]

Sometimes, we will express \( M \in F \) as a fn of \( X \) and \( Y \). We are then interested in cdfs of the form:

\[
F_Y(y|M(x,y)), M(x,y) \in F
\]

**Example:** Let \( M = \{x_1 < X \leq x_2\} \). Find \( F_Y(y|M) \)

\[
F_Y(y|x_1 < X \leq x_2) = \frac{P(\{Y \leq y\} \cap \{x_1 < X \leq x_2\})}{P(\{x_1 < X \leq x_2\})} = \frac{F_{X,Y}(x_2,y) - F_{X,Y}(x_1,y)}{F_X(x_2) - F_X(x_1)}
\]

We can differentiate this result to get

\[
f_Y(y|x_1 < X \leq x_2) = \frac{2}{2y} F_Y(y|x_1 < X \leq x_2) = \frac{\int_{x_1}^{x_2} \frac{d}{dx} F_{X,Y}(x,y) dx}{F_X(x_2) - F_X(x_1)}
\]
of particular interest is $f_p(y | x = x)$. To find this, take the previous result with $x_1 = x$, $x_2 = x + \Delta x$

which gives $f_p(y | x < x \leq x + \Delta x) = \frac{\int_{x}^{x+\Delta x} f_{x,y}(x,y) \, dx}{F_x(x+\Delta x) - F_x(x)}$

Then,

$$f_p(y | x = x) = \lim_{\Delta x \to 0} f_p(y | x < x \leq x + \Delta x)$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \frac{\int_{x}^{x+\Delta x} f_{x,y}(x,y) \, dx}{\frac{1}{\Delta x} \left( F_x(x+\Delta x) - F_x(x) \right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[ f(x+\Delta x, y) - f(x, y) \right]$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[ F_x(x+\Delta x) - F_x(x) \right]$$

where $f(x, y)$ is the antiderivative of $f_{x,y}(x,y)$ w.r.t. $x$.

and $\frac{\partial}{\partial x} f(x, y) = f_{x,y}(x, y)$

Thus, we have $f_p(y | x = x) = \frac{f_{x,y}(x,y)}{f_x(x)}$

Similarly, $f_x(x | y) = \frac{f_{x,y}(x,y)}{f_y(y)}$

We will sometimes use the notation

$$f_z(x | y) = f_z(x | y) = f(x | y)$$

when context makes clear the interest
Note: When $X$ and $Y$ are stat. indep. $f_{XY}(x,y) = f_X(x) f_Y(y)$

Then $f_Y(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = f_Y(y)$

and $f_X(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = f_X(x)$

Bayes' Relation

\[ f(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f(y|x)f_X(x)}{f_Y(y)} \]

Bayes' Theorem

Note that

\[ f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx = \int_{-\infty}^{\infty} f(y|x) f_X(x) \, dx \]

\[ \Rightarrow f(x|y) = \frac{f(y|x)f_X(x)}{\int_{-\infty}^{\infty} f(y|x) f_X(x) \, dx} \]
Summary of Bayes' Relation/Theorem

Bayes' Relation:

\[
P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}
\]

\[
P(A|\bar{B} = x) = \frac{f_2(x|A)}{f_2(x)} \cdot P(A)
\]

\[
f_2(x|y) = \frac{f_{2|y}(x,y)}{f_y(y)} \cdot f_2(x)
\]

Bayes' Theorem:

\[
P(A|B) = \frac{P(B|A) \cdot P(A)}{\sum_i P(B|A_i)P(A_i)}
\]

\[
f_2(x|A) = \frac{P(A|\bar{B} = x)}{\int_{-\infty}^{\infty} P(A|\bar{B} = x) \cdot f_2(x) \, dx}
\]

\[
f_2(x|y) = \frac{f_2(y|x)}{\int_{-\infty}^{\infty} f_2(y|x) \cdot f_2(x) \, dx}
\]
One Function of Two R.V.

Given two jointly dist. R.V. $X$ and $Y$ and a ftn. $g(x,y) : \mathbb{R}^2 \rightarrow \mathbb{R}$

We can form a new R.V. $Z = g(X,Y)$

How to calculate $F_Z(z)$ or $f_Z(z)$?

1. Let $D_z \subset \mathbb{R}^2$ (actually $D_z \subset \text{B}(\mathbb{R}^2)$) be defined as

   $D_z = \{ (x,y) : g(x,y) \leq z \}, \ z \in \mathbb{R}$

   $F_Z(z) = P(\{ Z \leq z \}) = P_Y(\{ (X,Y) \in D_z \})$

   $= \iint_{D_z} f_{XY}(x,y) \, dx \, dy = \int_{\mathbb{R}^2} f_{XY}(x,y) \, 1_{D_z}(x,y) \, dx \, dy$

   n.b. To determine $F_Z(z)$, we must carry out this procedure for each $z \in \mathbb{R}$. Difficulty depends on complexity of $g(x,y)$. Once we have $F_Z(z)$, we can get $f_Z(z) = \frac{dF_Z(z)}{dz}$

2. Use Auxiliary Variables and then use direct method for two R.V.s.

3. Monte Carlo Simulation
Example: \( g(x,y) = x+y \) \( \Rightarrow \) \( Z = X + Y \)

\[ F_Z(z) = P\{Z \leq z\} = P\{(X,Y) \in D_z\} \]

where \( D_z = \{(x,y): x+y \leq z\} \)

Thus \( F_Z(z) = \int \int_{D_z} f_{X,Y}(x,y) \, dx \, dy = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{3-y} f_{X,Y}(x,y) \, dx \right) \, dy \)

\[ \cdots \] (4)

• Now if \( X \) and \( Y \) are stat. indep.

\[ F_Z(z) = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{3-y} f_{X}(x) \, f_{Y}(y) \, dx \right) \, dy \]

\[ = \int_{-\infty}^{\infty} f_{Y}(y) \left( \int_{-\infty}^{3-y} f_{X}(x) \, dx \right) \, dy \]

\[ = \int_{-\infty}^{\infty} f_{Y}(y) \, F_Z(3-y) \, dy \]

and

\[ f_Z(z) = \frac{dF_Z(z)}{dz} = \frac{d}{dz} \left[ \int_{-\infty}^{\infty} f_{Y}(y) \, F_Z(3-y) \, dy \right] \]

\[ = \int_{-\infty}^{\infty} f_{Y}(y) \, f_Z(3-y) \, dy \leftarrow \text{Convolution Integral} \]

\[ = (f_{Y} * f_{X})(3) \]
Two Functions of Two RVs

Given two joint dist. RVs $X$ and $Y$ with pdf $f_{XY}(x,y)$

The two new RVs $Z = g(X,Y) \ (g: \mathbb{R}^2 \to \mathbb{R})$

$W = h(X,Y) \ (h: \mathbb{R}^2 \to \mathbb{R})$

How to find $f_{ZW}(z,w)$?

1. From the mapping analysis and definition of cdf/pdf

$$F_{ZW}(z,w) = P \{ (Z \leq z) \cap (W \leq w) \} = P \{ (X,Y) \in D_{zw} \}$$

Where $D_{zw} = \{ (x,y) : g(x,y) \leq z \text{ and } h(x,y) \leq w \}$

$$\Rightarrow F_{ZW}(z,w) = \iint_{D_{zw}} f_{XY}(x,y) \, dx \, dy$$

$$f_{ZW}(z,w) = \frac{\partial}{\partial z \partial w} F_{ZW}(z,w)$$

\textbf{NB:} Complexity of task depends on complexity of $g(x,y)$ and $h(x,y)$

2. Direct Joint Density Determination

3. Monte Carlo Simulation.
**Direct Joint Density Determination**

**Theorem:** Let $X$ and $Y$ be two jointly dist. RVs with pdf $f_{X,Y}(x,y)$, let $Z = g(X,Y)$ and $W = h(X,Y)$, and assume the functions $g(\cdot, \cdot)$ and $h(\cdot, \cdot)$ satisfy the following conditions:

(a) \[ Z = g(X,Y) \quad \frac{\partial X}{\partial Z} , \frac{\partial Y}{\partial Z} \quad \text{Can be uniquely solved} \quad \Rightarrow \quad X = X(Z,W) \quad W = Y(Z,W) \]

(b) \[ \frac{\partial X}{\partial Z} , \frac{\partial Y}{\partial Z} , \frac{\partial Y}{\partial W} , \frac{\partial Z}{\partial W} \quad \text{exist and are continuous} \]

Then, \[ f_{Z,W}(z,w) = f_{X,Y}(x(z,w), y(z,w)) \left| \frac{\partial (x,y)}{\partial (z,w)} \right| \]

where the Jacobian is given by the determinant

\[ \frac{\partial (x,y)}{\partial (z,w)} = \begin{vmatrix} \frac{\partial X}{\partial Z} & \frac{\partial X}{\partial W} \\ \frac{\partial Y}{\partial Z} & \frac{\partial Y}{\partial W} \end{vmatrix} = \frac{\partial X}{\partial Z} \frac{\partial Y}{\partial W} - \frac{\partial Y}{\partial Z} \frac{\partial X}{\partial W} \]

**Example** $X$ and $Y$ are zero-mean independent identically distributed (i.i.d.) Gaussian RVs with variance $\sigma^2$

\[ \Rightarrow f_{X,Y}(x,y) = \frac{1}{2\pi \sigma^2} \exp \left\{ -\frac{(x^2+y^2)}{2\sigma^2} \right\} \]

Let $R^2 = X^2 + Y^2$, \( \theta = \tan^{-1} \left( \frac{Y}{X} \right) \)

\[ \Rightarrow R = R \cos \theta \quad Y = R \sin \theta \]

Find $f_{R, \theta}(r, \theta)$
\[
\begin{align*}
\left\{ \begin{array}{l}
x = r \cos \theta \\
y = r \sin \theta
\end{array} \right. \\
\mathcal{f}_{r, \theta}(r, \theta) &= \mathcal{f}_{x, y}(r \cos \theta, r \sin \theta) \left| \frac{\partial (x, y)}{\partial (r, \theta)} \right| \\
&= \frac{1}{2\pi \sigma^2} \exp \left\{ -\frac{(r \cos^2 \theta + r \sin^2 \theta)}{2\sigma^2} \right\} \frac{1(r)}{2\sigma} \frac{1(\theta)}{2\pi}
\end{align*}
\]

\[
= \frac{r}{2\pi \sigma^2} \exp \left\{ -\frac{r^2}{2\sigma^2} \right\} \frac{1(r)}{2\sigma} \frac{1(\theta)}{2\pi}
\]
Auxiliary Variables

Sometimes we want to find pdf of one RV defined as a fn of two RVs: \( Z = g(Z, \Theta) \).
It is often easier to use the direct approach for \((X, \Theta) \rightarrow (Z, W)\). But we only have \( Z = g(Z, \Theta) \).
What do we do?

Introduce an auxiliary RV \( W = h(Z, \Theta) \). Then find \( f_{Z,W}(z,w) \). Finally, compute \( f_Z(z) = \int_{-\infty}^{\infty} f_{Z,W}(z,w) \, dw \).

How do we pick aux. RV \( W \)?

1. Pick it to make finding \( f_{Z,W}(z,w) \) easy
2. Pick it to make integration \( \int_{-\infty}^{\infty} f_{Z,W}(z,w) \, dw \) easy

A very popular choice \( W = X \) or \( W = \Theta \). This often works.

If \( Z = \sqrt{X^2 + \Theta^2} \), an aux. RV \( \Theta = \tan^{-1}(\frac{X}{\Theta}) \) works well.

In general, depends on problem and many require some guessing.  
Rect → Polar conv.
Mean (Expected) Value and Variance

Mean of \( Z = g(X, Y) \)

Theorem: Given two j-dist. RVs \( X \) and \( Y \) and a new RV \( Z = g(X, Y) \)

\[ E\{Z\} = E \{ g(X, Y) \} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{XY}(x, y) \, dx \, dy \]

Proof: Let \( \Delta Z = \{ (x, y) : z < g(x, y) < z + \Delta z \} \)

To each differential element in

\[ E\{Z\} = \int_{-\infty}^{\infty} z f(z) \, dz \]

There corresponds a \( \Delta Z \) in the \( x, y \) plane.

As \( \Delta z \) covers the whole real line, the corresponding \( \Delta Z \) cover the whole \( x, y \) plane and are not overlapping. Hence

\[ \int_{-\infty}^{\infty} z f(z) \, dz = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{XY}(x, y) \, dx \, dy \]

Note: Let \( X \) and \( Y \) be discrete RVs such that

\[ P_{kj} = P(\{ X = x_j \} \land \{ Y = y_k \}) \]

\[ f_{XY}(x, y) = \sum_j \sum_k P_{kj} \delta(x - x_j, y - y_k) \]

\[ \Rightarrow E\{ g(X, Y) \} = \sum_j \sum_k P_{jk} \cdot g(x_j, y_k) \]
Conditional Expectation

\[ E[Z|Y] = E\{g(x,Y)\} \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y|Y) \, dx \, dy \]

- Specific case, \( g(x,y) = x, M = \{Y = y\} \)
\[ E\{X|\{Y = y\}\} = \int_{-\infty}^{\infty} x f_{X}(x|Y = y) \, dx \]
\[ = \int_{-\infty}^{\infty} x f(x|y) \, dx \]

where \( f(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} \)

Iterated Expectation

\[ E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y) \, dx \, dy \]
\[ = \int_{-\infty}^{\infty} g(x,y) f_Y(y|x) f_X(x) \, dx \, dy \]
\[ = \int_{-\infty}^{\infty} f_X(x) \left[ \int_{-\infty}^{\infty} g(x,y) f_Y(y|x) \, dy \right] \, dx \]

\[ E[g(X,Y)]_{X=x} \text{ is a fn of } x \]

\[ \therefore E[Z] = E[g(X,Y)] = E_X \{ E_Y \{ g(X,Y) | X \} \} \]

\[ (E[Y] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f_{XY}(x,y) \, dx \, dy = E_X \{ E_Y \{ Y | X \} \} ) \]

Iterated Expectation.
**Linearity of Expectation**

\[ E \left( \sum_{i=1}^{n} \alpha_i g_i(x, y) \right) = \sum_{i=1}^{n} \alpha_i E \{ g_i(x, y) \} \]

In particular
\[ E \{ x + y \} = E \{ x \} + E \{ y \} \]
\[ E \{ g(x) + h(y) \} = E \{ g(x) \} + E \{ h(y) \} \]

**Variance of** \( Z = g(x, y) \)

\[ \text{Var} \{ Z \} = E \{ (Z - \mu)^2 \} = E \{ \text{Var} \{ Z | Y \} \} + \text{Var} \{ E \{ Z | Y \} \} \]

**Proof**

\[ \text{Var} \{ Z \} = E \{ Z^2 \} - (E \{ Z \})^2 \]
\[ = E \{ E \{ Z^2 | Y \} \} - (E \{ E \{ Z | Y \} \})^2 \]
\[ = E \{ E \{ Z^2 | Y \} \} - E \{ E \{ Z | Y \} \}^2 \]
\[ + E \{ E \{ Z | Y \}^2 \} - (E \{ E \{ Z | Y \} \})^2 \]
\[ = E \{ E \{ Z^2 | Y \} - E \{ Z | Y \}^2 \} + \text{Var} \{ E \{ Z | Y \} \} \]
\[ = E \{ \text{Var} \{ Z | Y \} \} + \text{Var} \{ E \{ Z | Y \} \} \]
Joint Moments

**Correlation**

Deﬁnition: Given two j-distr. RVs $X$ and $Y$, the correlation between them is

$$\text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{E\{XY\} - E\{X\}E\{Y\}}{\sigma_X \sigma_Y}$$

Covariance

Their covariance is

$$\text{Cov}(X, Y) = E\{(X - \bar{X})(Y - \bar{Y})\}$$

**Correlation Coefficient**

and their correlation correlation is

$$\rho_{XY} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{E\{X\}E\{Y\}}{\sigma_X \sigma_Y}$$

n.b. $-1 \leq \rho_{XY} \leq 1$

$$\left|E[f(X,Y)]\right| \leq \sqrt{E[f^2(X)] E[f^2(Y)]} \quad \text{(Schwarz Inequality)}$$

**Fact:** If $X$ and $Y$ are stat. indep. then $\rho_{xy} = 0$

The converse is not generally true.

**Specially Case:** If $X$ and $Y$ are jointly Gaussian and $\rho_{xy} = 0$

Then $X$ and $Y$ are stat. indep.
**Defn**: Two RV, \( Z \) and \( Y \) are said to be **uncorrelated** if their covariance is zero. This is true if any of the following hold:

1. \( \text{COV}(Z,Y) = 0 \)
2. \( Z \perp Y = 0 \)

These are equivalent conditions.

**Theorem**: If two RV, \( Z \) and \( Y \) are stat. indep, they are uncorrelated.

**Proof**: Simple, must show \( Z \perp Y \Rightarrow E[ZY] = E[Z]E[Y] \)

**N.b.** The converse is generally true

\[
E[ZY] = E[Z]E[Y] \quad \Rightarrow \quad Z \perp Y
\]

**Counter Example**: Let \( Z \) be a R.V. with even pdf \( f_Z(x) = f_Z(-x) \), \( \forall x \). And let \( Y = Z^2 \)

**Claim**: \( E[ZY] = 0 \). But knowledge of \( Z \) completely determine \( Y \). So \( Z \) and \( Y \) are not stat. indep.

\[
E[ZY] = E[Z(Z^2)] = E[Z^3] = \int_{-\infty}^{\infty} x^3 f_Z(x) dx = 0
\]
**Defn:** The joint (Non central) moments of \( \mathbf{X} \) and \( \mathbf{Y} \) are

\[
M_{jk} = E \{ x^j y^k \} = \iint_{\mathbb{R}^2} x^j y^k f_{\mathbf{X}\mathbf{Y}}(x,y) \, dx \, dy
\]

and the joint central moments are defined as

\[
U_{jk} = E \{ (\mathbf{X} - \overline{\mathbf{X}})^j (\mathbf{Y} - \overline{\mathbf{Y}})^k \}
= \iint_{\mathbb{R}^2} (x - \overline{\mathbf{X}})^j (y - \overline{\mathbf{Y}})^k f_{\mathbf{X}\mathbf{Y}}(x,y) \, dx \, dy
\]

**Note,**

\[
E \{ aX + bY \} = aE\{X\} + bE\{Y\}
\]

\[
\text{Var} \{ aX + bY \} = a^2 \text{Var} \{ X \} + b^2 \text{Var} \{ Y \} + 2ab \text{Cov} \{ X, Y \}
\]