§ Set Theory

Why set theory? The best way to describe a random experiment.

e.g. Roll a die:

\[ \mathcal{S} = \{1, 2, 3, 4, 5, 6\} \]

We can define events:

\[ A_1 = \text{Outcome is odd} = \{1, 3, 5\} \]
\[ A_2 = \text{Outcome is divisible by 3} = \{3, 6\} \]
\[ A_3 = \text{Outcome is prime} = \{2, 3, 5\} \]

n.b. Each event is described as a subset of \( \mathcal{S} = \{1, 2, 3, 4, 5, 6\} \).

* There are \( 2^6 = 64 \) distinct subsets of \( \mathcal{S} \).
* To characterize the random experiment, we must know the probability of each event.
Events

\[ \mathcal{F}(\Omega) = \{ A_1, A_2, \ldots, A_{64} \} \text{ where } A_k \subseteq \Omega \]

The set of all events

* Our random experiment is completely characterized by \((\Omega, \mathcal{F}(\Omega), \mathbb{P}(\cdot))\)

where \( \mathbb{P}(\cdot) : \mathcal{F}(\Omega) \rightarrow [0, 1] \)

\( \mathbb{P}(\cdot) \) assigns "probability" to the events.

Elements of Set Theory

* A set is a collection of objects
* Each object in a set is called an element, point, or member.
* \( w \) is an element of a set in \( \mathbb{A} \), \( w \in \mathbb{A} \)
* \( w \) is not in \( \mathbb{A} \), \( w \notin \mathbb{A} \).

Two common ways to specify sets:

1. Listing the elements
   \[ A = \{ 1, 2, 3, 4, 5, 6 \} \]

2. Specifying a property common to all elements
   \[ A = \{ w \in \mathbb{Z} : 0 < w < 7 \} \]
Defn: Two sets \( A \) and \( B \) are equal iff they contain exactly the same elements. We then write \( A = B \). If \( A \) and \( B \) are not equal, we write \( A \neq B \).

Defn: Of every element of a set \( A \) is also an element of set \( B \), we say that \( A \) is a subset of \( B \), and write \( A \subseteq B \).

Defn: The set with no elements is called the empty set, and is denoted \( \emptyset \) or \( \{ \} \).

Defn: In a given problem, the set containing all possible elements of interest is called the universal set, or space.

* We denote this set by \( \mathcal{S} \) or \( \Omega \).
  * All sets of interest will be subsets of \( \mathcal{S} \).
  * For any set \( A \) considered:

\[ \emptyset \subseteq A \subseteq \mathcal{S} \]
Set Operations:

1. Intersection: $A \cap B = \{ w \in S : w \in A \text{ and } w \in B \}$

2. Union: $A \cup B = \{ w \in S : w \in A \text{ or } w \in B \}$

3. Complement: $\overline{A} = \{ w \in S : w \notin A \}$
4. Set Difference: \( A - B = \{ w \in S : \text{WEA and } w \notin B \} \)

5. Symmetric Difference:
   \[ A \Delta B = \{ w \in S, \text{ WEA or } w \in B, \text{ but not both} \} \]
   \[ = (A \cup B) - (A \cap B) \]

**Fact:** Two sets \( A \) and \( B \) are equal iff \( A \subseteq B \) and \( B \subseteq A \).
Algebra of Sets:

1. \( A \cup B = B \cup A \)  \{ \text{Commutative law of } \cup \}
2. \( A \cap B = B \cap A \)
3. \( A \cup (B \cup C) = (A \cup B) \cup C \) \{ \text{Associative law of } \cup \}
4. \( A \cap (B \cap C) = (A \cap B) \cap C \)
5. \( A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \) \{ \text{Dist. } \cap \text{ over } \cup \}
6. \( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \)
7. \( \overline{\overline{A}} = A \)
8. \( \overline{A \cup B} = \overline{A} \cap \overline{B} \) \{ \text{Demorgan's Law} \}
9. \( \overline{A \cap B} = \overline{A} \cup \overline{B} \)
10. \( \overline{\overline{A}} = A \)
11. \( A \cup \emptyset = A \)
12. \( A \cap \emptyset = \emptyset \)
13. \( A \cup \emptyset = A \)
14. \( A \cup \emptyset = \emptyset \)
15. \( A \cup \overline{A} = \emptyset \)
16. \( A \cap \overline{A} = \emptyset \)

\text{\textit{n.b.}} \text{ of in a set identity, we replace all unions by intersections all intersections by unions, all the sets } \emptyset \text{ and } \emptyset \text{ by } \emptyset \text{ and } \emptyset, \text{ the identity preserved.}

"Duality Principle"
Index Collection of Sets

\[ \big\{ A_i, \ i \in I \big\} \]

Where \( I \) is an index set.
This is a "set of sets".
There is one set \( A_i \) for each \( i \in I \).

Typical index sets \( I \):
- \( \mathbb{N} = \{ 1, 2, 3, \ldots \} \)
- \( \mathbb{Z} = \{ \ldots, -1, 0, 1, \ldots \} \)
- \( \mathbb{Z}^+ = \{ 0, 1, 2, \ldots \} \)
- \( \mathbb{I}_n = \{ 0, 1, 2, \ldots, n-1 \} \) finite index set
- \( \mathbb{R} = (-\infty, +\infty) \) the real line

**Defn**: An index sets is **finite** if it has a finite number of elements. e.g. In above

**Defn**: An index set is **countable** if it has an infinite number of element and the element can be put in 1-1 correspondence with the natural number \( 1, 2, 3, \ldots \).
The intervals:

\((a, b) = \{ x \in \mathbb{R}, a < x < b \} \)
\((a, b] \)
\([a, b) \)
\([a, b] \)

are all uncountable sets.

**Defn:** Given an indexed family of sets \(\{A_i, i \in I\}\),
the union of the family is \(\bigcup_{i \in I} A_i = \{ w : \exists i \in I \text{ with } w \in A_i \}\),
for at least one \(i \in I\),
and the intersection
is \(\bigcap_{i \in I} A_i = \{ w : w \in A_i \text{ for all } i \in I \}\).

**Defn:** A family of sets \(\{A_i, i \in I\}\) is disjoint, if
\(A_i \cap A_j = \emptyset, \forall i, j \in I, i \neq j\).

**Defn:** A family of sets \(\{A_i, i \in I\}\) is collectively exhaustive if \(\bigcup_{i \in I} A_i = \mathbb{R}\).

**Defn:** A family of sets \(\{A_i, i \in I\}\) is called a partition of the space if it is both disjoint and collectively exhaustive.
**Defn.** A family \( \{A_i : i \in I\} \) is a partition of \( G \subseteq \delta \) if it is disjoint and \( \bigcup_{i \in I} A_i = G \).

**Fact.** Let \( \{A_i : i \in I\} \) be a partition of \( \delta \), and let \( B_i = A_i \cap G \), \( i \in I \), \( G \subseteq \delta \). Then \( \{B_i : i \in I\} \) is a partition of \( G \).

**Proof.** (i) \( \{B_i : i \in I\} \) disjoint

(ii) \( \bigcup_{i \in I} B_i = G \).