Introduction to Reliability Analysis

Let $S =$ the Load (or Demand)

Let $R =$ the Resistance (or Capacity)

Both $S$ and $R$ are random in nature, characterized by probability density function as shown in Figure 1.

![Probability density function diagram](image)

**Figure 1. Fundamentals of Risk Evaluation**

If the maximum demand exceeds the minimum capacity, the distributions will overlap and there will be a nonzero probability failure. The overlap between the two curves (the shaded region in Figure 1) provides a qualitative measure of the probability of failure. This area of overlap depends on three factors:

1. The relative positions of the two curves.
2. The dispersion of the two curves.
3. The shapes of the two curves.
**Deterministic Approach**

The principal aim of engineering design is the assurance of system performance (within the constraint of economy). Conventional design approaches achieve this objective by shifting the positions of the curves through the use of safety factors.

**Central Safety Factor**

\[
FS_{central} = \frac{\mu_r}{\mu_s}
\]

**Nominal Safety Factor**

\[
FS_{nominal} = \frac{R_N}{S_N}
\]

The nominal resistance (or capacity) \(R_N\) is usually a conservative value, perhaps one, two, or three standard deviations below the mean value. The nominal load (or demand) \(S_N\) is also a conservative value; however, it is several standard deviations above the mean value.

**Factor Safety Design**

\[
\mu_r \geq FS_{central} \cdot \mu_s
\]

\[
R_N \geq FS_{nominal} \cdot S_N
\]

**Load and Resistance Factor Design**

The factor of safety is to account for the combination of uncertainties of the load and resistance. In load and resistance factor design (LRFD) concept, the capacity reduction factor (generally less than one) and load factors (generally more than one) are used to achieve the same objective.

\[
\bar{\phi} \mu_r \geq \bar{\eta} \mu_s
\]
\( \phi R_N \geq \gamma S_N \)

In structural design, where the uncertainty of dead load and live load are different, the LRFD is modified as

\[ \phi R_N \geq \gamma_D D + \gamma_L L \]

**Probabilistic Approach**

Through regulation or tradition, the same value of safety factor is often applied to conditions that involve widely varying degrees of uncertainty (i.e. dispersion). This is not logical. A more rational approach would be to compute the risk by accounting for all three overlap factors, and selecting the design variables so that an acceptable risk of failure is achieved. This is the foundation of the risk-based design concept.

**Performance function**

\[
g(R, S) = \begin{cases} 
> 0 & \text{safe state} \\
= 0 & \text{limit state} \\
< 0 & \text{failure state}
\end{cases}
\]

In reality, R and S are functions of many other design parameters \( X_1, X_2, \ldots, X_n \). The performance function can be written as

\[
g(X_1, X_2, \ldots, X_n) = \begin{cases} 
> 0 & \text{safe state} \\
= 0 & \text{limit state} \\
< 0 & \text{failure state}
\end{cases}
\]

or in vector form,

\[
g(X) = \begin{cases} 
> 0 & \text{safe state} \\
= 0 & \text{limit state} \\
< 0 & \text{failure state}
\end{cases}
\]
Probability of Failure

\[ P_f = \int_{g(X) < 0} f_X(x) \, dx \]

If \( X_i \)'s are statistically independent,

\[ P_f = \int_{g(X) < 0} f_{X_1}(x_1) \cdots f_{X_n}(x_n) \, dx_1 \cdots dx_n \]

Conceptually easy, but realistically difficult to evaluate except for special cases (such as Gaussian design variable and linear performance function) because

1. Impractical to construct \( f_X(x) \) on the existing database.
2. Multidimensional integration over the generally irregular domain \( g(X) < 0 \) is formidable.

One feasible approach is assuming independence and approximating each design variable with common pdf and then conducting the multidimensional integration over the irregular \( g(X) < 0 \) using Monte Carlo integration methods. The computation involved is this approach is typically intensive.

With this approach, however, the information on the shape of the probability density function of the resistance and loads is usually difficult to obtain, and engineers must formulate an acceptable design methodology using only the information on means and standard deviations.

Reliability Index

Define reliability index as

\[ \beta = \frac{\mu_s}{\sigma_g} \]
When the design variables are Gaussian variables and the performance function is linear, the reliability index is directly related to the probability of failure.

\[ P_f = 1 - \Phi(\beta) \]

![Figure 2.](image)

Under most other cases, the reliability index is not directly related to probability of failure. However, it does provide a measure of reliability of an engineering system; and it permits comparison of reliability among different structures or modes of performance without having to calculate absolute probability values. It can also be used to evaluate probability of failure approximately (linear and Gaussian approximation).
Mean and standard deviation of performance function can be evaluated by the sample statistics of Monte Carlo simulation or by direct integration using Monte Carlo integration. With these approach, however, the information on the shape of the probability density function of the design variables usually difficult to obtain, and engineers must formulate an acceptable design methodology using only the information on means and standard deviations. There are several ways to evaluate reliability index using only the information on means and standard deviations:

**Point Estimate Method**
- $2^n$ method
- $2n$ methods

**Taylor Series Method**
- First-Order Reliability Methods (FORM)
  - Mean Value First Order Second Moment (MVFOSM)
    Taylor series expansion at mean-value point.
  - Advanced First Order Second Moment Method (AFOSM)
    Taylor series expansion at design point.
- Second-Order Reliability Methods (SORM)

**Reliability Assessment**
The design of a system is given by a set of values of parameters characterizing the system, and by a set of pertinent safety relations. In this way one identifies the safe and unsafe regions in the space of random variables relevant to the problem as modeled in terms of random load and resistance. A *design point* is usually defined as the most probable point on the failure boundary or limiting state of interest between the safe and unsafe regions, or as some conditioned point on the boundary. Similarly,
a characteristic point may be defined in the domain of random variables by assigning to each variable a characteristic (average or cautious) value. Quotients between the design and characteristic values (for load, and the opposite for resistance) represent the partial safety factors. There are different level of reliability assessment depending on the extent of considerations on the factors that affects the probability of failure.

**Level 1 Reliability Assessment**

Level 1 Reliability Assessment, which is provided by a number of partial safety factors, or safety margins related to characteristic values of variates, is the most popular method for design practice. In most cases these safety factors are not explicitly related to the probability distribution of random variables or to the failure probability, but they reflect a standard variability and tolerance risk. Only if the safety factors are explicitly related to the failure probabilities and to the joint pdf of the involved variates does a Level 1 assessment give explicit information on risk.

**Level 2 Reliability Assessment**

Level 2 reliability assessment is performed by calculating the reliability index and assuming the variates to be jointly normal/lognormal distributed, and the shape of the failure boundary to be approximated by a linear or circular surface. This method yields the probability of failure, the design point, partial safety factors, and the relative importance of the uncertainty of the single variates in estimating the probability of failure. Its accuracy depends on the combined effect of its assumptions to represent the limiting state of interest and the joint pdf of the variates. This approximation is conservative if the exact nonlinear safe-unsafe boundary is convex toward the origin, but it is nonconservative for a concave boundary.
Level 3 Reliability Assessment

Level 3 reliability assessment requires the evaluation of the probability of failure by integrating over the unsafe region of the joint pdf of the involved variates. This method is conceptually sound and sometimes called the exact method although some approximations are used for integration.
Special Cases

The CDF/PDF of simple performance functions and hence the probability of failure can be determined analytically in terms of CDF/PDF of design variables (typically involves integration). For a certain type of CDF/PDF of design variables, the CDF/PDF of a simple performance function can be easily obtained; and the reliability index can be expressed in a very neat form.

**Safety Margin**

\[ M = R - S \]

**CDF of Safety Margin**

\[ F_M (m) = P\{M < m\} = P(\{(R, S) \in D_M\} \]

where \[ D_M = \{(R, S) : R - S \leq m\} \]

![Figure 1](image)

Thus,

\[ F_M (m) = \int_{D_M} f_{R,S} (r,s)drds = \int_{-\infty}^{m} \int_{-\infty}^{m-r} f_{R,S} (r,s)drds \]

Now if R and S are statistically independent,

\[ F_M (m) = \int_{-\infty}^{m} \int_{-\infty}^{m-r} f_R (r) f_S (s)drds = \int_{-\infty}^{m} \int_{-\infty}^{m-r} f_R (r) df_s (s)ds \]

\[ = \int_{-\infty}^{m} F_R (m + s) f_s (s)ds \]
**Probability of Failure**

\[ P_f = F_M(0) = \int_{-\infty}^{\infty} F_R(s)f_S(s)ds \]

**Reliability Index (defined by performance function, \( g(-) = M \))**

Let the performance function be the safety margin. If \( R \) and \( S \) are statistically uncorrelated, then

\[ \mu_M = \mu_R - \mu_S \]

\[ \sigma_M = \sqrt{\sigma_R^2 + \sigma_S^2} \]

\[ \beta = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} \]

And if \( R \) and \( S \) are normally distributed in addition to the uncorrelated property, probability of failure is related to reliability index directly as

\[ P_f = 1 - \Phi(\beta) \]

**Relationship between Safety Factor and Reliability Index**

To eliminate the square root sign to separate \( R \) and \( S \), a parameter \( \varepsilon \) can be introduced as

\[ \varepsilon = \frac{\sqrt{\sigma_R^2 + \sigma_S^2}}{\sigma_R + \sigma_S} \]

\( \varepsilon \) can be considered to be approximately 0.75 in most cases. The reliability index becomes

\[ \beta = \frac{\mu_R - \mu_S}{\varepsilon(\sigma_R + \sigma_S)} \]

After the variables are separated, the equation becomes

\[ \mu_R - \varepsilon \beta \sigma_R = \mu_S - \varepsilon \beta \sigma_S \]

or

\[ \mu_R (1 - \varepsilon \beta \delta_R) = \mu_S (1 - \varepsilon \beta \delta_S) \]

Where \( \delta_R = \frac{\sigma_R}{\mu_R} \) and \( \delta_S = \frac{\sigma_S}{\mu_S} \) are coefficients of variation.
\[ FS_{\text{central}} = \frac{1 + \varepsilon \beta_z \delta_z}{1 - \varepsilon \beta_r \delta_r} \]

We can express the capacity reduction factor corresponding to the central safety factor as
\[ \overline{\phi} = 1 - \varepsilon \beta \delta_r \]
\[ \overline{\gamma} = 1 + \varepsilon \beta \delta_z \]

To define the nominal safety factor, the nominal or characteristic values of the load and resistance need to be introduced as
\[ R_N = \mu_r (1 - k_r \delta_r) \]
\[ S_N = \mu_z (1 - k_z \delta_z) \]

The conventional or nominal safety factor becomes
\[ FS_{\text{nominal}} = \left( 1 + \varepsilon \beta_z \delta_z \right) \frac{1 - k_r \delta_r}{1 + k_z \delta_z} \]

Thus, after the variables are separated, the nominal capacity reduction factor and the load factor can be shown to be
\[ \phi = \frac{1 - \varepsilon \beta_r \delta_r}{1 - k_r \delta_r} \]
\[ \gamma = \frac{1 + \varepsilon \beta_z \delta_z}{1 + k_z \delta_z} \]

**Safety Factor**
\[ Y = \frac{R}{S} \]

**CDF of Safety Factor**
Let \( Y = R/S \), and \( Z = S \)
\[ \frac{\partial (R, S)}{\partial (Y, Z)} = \begin{vmatrix} \frac{\partial R}{\partial Y} & \frac{\partial R}{\partial Z} \\ \frac{\partial S}{\partial Y} & \frac{\partial S}{\partial Z} \end{vmatrix} = \begin{vmatrix} S & \frac{\partial R}{\partial Z} \\ 0 & 1 \end{vmatrix} = S = Z \]
\[ f_{Y,Z}(y,z) = f_{R,S}(r(y,z),s(y,z)) \cdot \frac{\partial (r,s)}{\partial (Y,Z)} \]
\[ = f_{R,S}(yz,z) \cdot \frac{\partial (r,s)}{\partial (Y,Z)} \]
\[ f_Y(y) = \int_0^\infty f_{R,S}(yz, z)dz = \int_0^\infty f_{R,S}(ys, s)ds \]

Now if R and S are statistically independent,
\[ F_Y(y) = \int_0^\infty \left[ \int_0^\infty f_R(ys)f_S(s)ds \right]dy = \int_0^\infty \left[ \int_0^\infty f_R(ys)dy \right]f_S(s)ds \]
\[ = \int_0^\infty \frac{1}{y}F_R(ys)f_S(s)ds \]

**Probability of Failure**
\[ P_f = F_M(1) = \int_{-\infty}^{\infty} F_S(s)f_S(s)ds \]

**Reliability Index (defined by performance function, \(g(\cdot) = \ln Y\))**
Let the performance function be the log of safety factor.
\[ \ln Y = \ln R - \ln S \]

If R and S are log-normal variables and statistical independent, with mean and standard deviation \((\mu_R, \sigma_R)\) and \((\mu_S, \sigma_S)\), then \(\ln R\) and \(\ln S\) become normally distributed with mean and standard deviation \((\lambda_R, \xi_R)\) and \((\lambda_S, \xi_S)\), where
\[ \lambda = \ln \mu - \frac{1}{2} \xi^2 \]
\[ \xi^2 = \ln \left(1 + \sigma^2 / \mu^2 \right) = \ln \left(1 + \delta^2 \right) \]

then,
\[ \mu_{\ln Y} = \lambda_R - \lambda_S \]
\[ \sigma_{\ln Y} = \sqrt{\xi_R^2 + \xi_S^2} \]

\[ \beta = \frac{\lambda_R - \lambda_S}{\sqrt{\xi_R^2 + \xi_S^2}} = \frac{\ln \left[ \frac{\mu_R}{\mu_S} \right]}{\sqrt{\ln \left[ \frac{1 + \delta_R}{1 + \delta_S} \right]}} \]

If \(\delta_R\) and \(\delta_S\) are not large, say \(\leq 0.30\), \((1 + \delta^2) = \delta^2\), reliability index can be simplified as
\[ \beta = \frac{\ln \left[ \frac{\mu_R}{\mu_S} \right]}{\sqrt{\delta_R^2 + \delta_S^2}} \]

And probability of failure is related to reliability index directly as

\[ P_f = 1 - \Phi(\beta) \]

**Relationship between Safety Factor and Reliability Index**

The parameter \( \varepsilon_i \), similar to parameter \( \varepsilon \) can be introduced as

\[ \varepsilon = \frac{\sqrt{\ln(1 + \sigma_R^2) + \ln(1 + \sigma_S^2)}}{\sqrt{\ln(1 + \sigma_R^2) + \ln(1 + \sigma_S^2)}} \]

The capacity reduction factor and load factor corresponding to the central safety factor can be expressed as

\[ \phi = \frac{\exp[-\beta \varepsilon_i \sqrt{\ln(1 + \delta_R^2)}]}{\sqrt{\ln(1 + \delta_R^2)}} \]

\[ \gamma = \frac{\exp[-\beta \varepsilon_i \sqrt{\ln(1 + \delta_S^2)}]}{\sqrt{\ln(1 + \delta_S^2)}} \]

To obtain these factors with respect to the nominal safety factor, the nominal values of the resistance can be shown to be

\[ \ln R_N = \ln \mu_r - k_r \delta_R \]

or

\[ r_N = \mu_r \exp(-k_r \delta_R) \]

Similarly,

\[ S_N = \mu_s \exp(k_s \delta_S) \]

The conventional or nominal safety factor becomes

\[ FS_{\text{nominal}} = \left( \frac{\mu_r}{\mu_s} \right) \left( \frac{\exp(-k_r \delta_R)}{\exp(k_s \delta_S)} \right) \]

The corresponding nominal capacity reduction factor and load factor are

\[ \phi = \Phi \exp(k_s \delta_S) \]

\[ \gamma = \Phi \exp(-k_s \delta_S) \]
**Geometric Interpretation of Reliability Index**

For two uncorrelated random variables, R and S, the reliability index defined by the performance function, \( g(-) = R - S \), equals to the distance of the failure line to the origin of the reduced (or standardized) variables, \( R' \) and \( S' \). And the corresponding point is called the design point. If R and S are also normally distributed, the design point is the point of maximum likelihood of the limit state.

![Diagram](image)

**Figure 7.4 Hasofer–Lind Reliability Index: Linear Performance Function**