**Stochastic Process**

The idea of a stochastic process is a straightforward extension of that of a RV.

Instead of mapping each \( w \in S \) to a number \( \mathcal{X}(w) \), we map it to a function of time \( \mathcal{X}(t,w) \).

\[ \begin{array}{c}
\mathcal{X}(w_1) \\
\mathcal{X}(w_2) \\
\mathcal{X}(w_k) \\
\end{array} \quad \begin{array}{c}
\mathcal{X}(t,w_1) \\
\mathcal{X}(t,w_2) \\
\mathcal{X}(t,w_k) \\
\end{array} \]

n.b. There is nothing random about the individual sample functions. Randomness results from outcome \( w \in S \).
Note that if we pick a particular \( t = t_0 \), \( \mathcal{X}(t_0, w) \) is just a RV.

**Defn.** A Stochastic Process, or random process, defined on \((S, \mathcal{F}, \mathbb{P})\) is a family of random variable \( \{\mathcal{X}(t); t \in T\} \) indexed by \( t \), where the index set \( T \) can be discrete or continuous.
Notes:

1. If $T$ is a continuous subset of $\mathbb{R}$, $X(t)$ is called a continuous-time random process.

2. If $T$ is a discrete set, (usually the integer or $\mathbb{N}$) $X(t)$ is called a discrete-time random process.

3. $X(t)$ is called a discrete-state random process if for all $t \in T$, it takes on values from a discrete set. Otherwise it is a continuous-state R.P.

Examples:

ex1: Random vector as a discrete-time R.P.

ex2: A binary signal that changes its value at random points in time.

(continuous-time discrete-state R.P.)

ex3: Random Noise Waveform

ex4 Brownian Motion: Motion of a particle along axis as a function of time due to molecular collisions

[A. Einstein, 1906]
Prob. Description of R.P.s.

We will use cdfs and pdfs to characterize R.P.s. R.P. evaluated at any particular time is a RV. We can write down its pdf and cdf.

**Defn.** The first-order cdf of a R.P. \( Z(t) \) is
\[
F_{Z(t)}(x; t) = P(\{Z(t) \leq x\})
\]
The first-order pdf is
\[
f_{Z(t)}(x; t) = \frac{dF_{Z(t)}(x; t)}{dx}
\]

First-order cdf useful for considering \( Z(t) \) at single time instance. But we are often interested in interaction of values of \( Z(t) \).

**Defn.** The n-th order cdf of R.P. \( Z(t) \) is
\[
F_{Z(t)}(x_1, \ldots, x_n; t_1, \ldots, t_n)
\]
= \( P(\{Z(t_1) \leq x_1\} \cap \ldots \cap \{Z(t_n) \leq x_n\}) \)

The n-th order pdf is
\[
f_{Z(t)}(x_1, \ldots, x_n; t_1, \ldots, t_n) = \frac{dF_{Z(t)}(x_1, \ldots, x_n; t_1, \ldots, t_n)}{dx_1, \ldots, dx_n}
\]

of particular importance are the second-order (n=2) cdf and pdf.
Notation: We will use the notation \( \mathbb{X}(t) \) to designate
a R.R. Technically we should write \( \mathbb{X}(t,w) \) where
\( \mathbb{X}(\cdot, \cdot) : T \times \mathcal{E} \rightarrow \mathbb{R} \)

The notation \( \mathbb{X}(t) \) is used to represent the following entities:

1. \( \mathbb{X}(t,w) \) is a family of functions, here \( t \in T \)
    and \( w \in \mathcal{E} \) are variable.

2. It is a single function of time \( \mathbb{X}(t,w_0) \) for
    a fixed \( w_0 \in \mathcal{E} \). These single functions are
    called sample functions.

3. If \( t = t_0 \) is fixed and \( w \in \mathcal{E} \) is the outcome
    of a random experiment \( \mathbb{X}(t_0) = \mathbb{X}(t_0,w) \) is
    a random variable.

4. If \( w = w_0 \), and \( t = t_0 \), are fixed, then \( \mathbb{X}(t_0, w_0) \)
    is a number.
Properties and Descriptions of RPs.

A complex R.P.
\[ Z(t) = \mathcal{X}(t) + i \mathcal{Y}(t) \]
where \( \mathcal{X}(t) \) and \( \mathcal{Y}(t) \) are real RPs, is completely characterized by
\[ F(x_1, \ldots, x_n, y_1, \ldots, y_m; t_1, \ldots, t_n) \]
\[ \mathcal{X}(t_i), \mathcal{X}(t_n), \mathcal{Y}(t_i), \ldots, \mathcal{Y}(t_n), \]
for all \( n \) and all \( t_1, \ldots, t_n \).

**Defn:** For a RP \( \mathcal{X}(t) \), real or complex

- **Mean:** \[ \mathcal{E}(t) = E\{\mathcal{X}(t)\} \]
- **Auto correlation:** \[ R_{\mathcal{X}\mathcal{X}}(t_1, t_2) = E\{\mathcal{X}(t_1)\mathcal{X}^*(t_2)\} \]
  
  n.b. \( R_{\mathcal{X}\mathcal{X}}(t_2, t_1) = E\{\mathcal{X}(t_2)\mathcal{X}^*(t_1)\} = E\{\mathcal{X}(t_1)\mathcal{X}^*(t_2)\}^* \)
  
  \[ = R_{\mathcal{X}\mathcal{X}}^*(t_1, t_2) \]
- **Auto covariance:**
  \[ C_{\mathcal{X}\mathcal{X}}(t_1, t_2) = E\{\mathcal{X}(t_1) - \mathcal{E}(t_1)\mathcal{X}(t_2) - \mathcal{E}(t_2)\mathcal{X}^*(t_1)\} \]
  
  \[ = R_{\mathcal{X}\mathcal{X}}(t_1, t_2) - \mathcal{E}(t_1)\mathcal{E}(t_2) \]

The variance of \( \mathcal{X}(t) \) is given by \( \text{Var}[\mathcal{X}(t)] = C_{\mathcal{X}\mathcal{X}}(t, t) \)
- **Cross-correlation**

\[ R_{XY}(t_1, t_2) = E\{ X(t_1) Y^*(t_2) \} = R_{YX}(t_2, t_1) \]

- **Cross-covariance**

\[ C_{XY}(t_1, t_2) = E\{ (X(t_1) - \mu_X(t_1))(Y(t_2) - \mu_Y(t_2))^* \} \]

\[ = R_{XY}(t_1, t_2) - \mu_X(t_1) \mu_Y^*(t_2) \]

**Important Examples**

- **White-noise process.**

  A RP \( W(t) \) is called a white-noise process if

  \[ C_{WW}(t_1, t_2) = 0, \ t_1 \neq t_2 \]

- **Gaussian RP**

  \( X(t) \) is called Gaussian RP if the RVs \( X(t_1), \ldots, X(t_n) \)

  are jointly Gaussian. \( \forall n, \forall t_1, t_2, \ldots, t_n \)
Stationary Random Process.

- These are processes for which (roughly speaking) a change in time origins does not affect the probabilistic description of the process.

- Stationary RPs model many real world phenomenon accurately.

A RP $Z(t)$ is called stationary or strict-sense stationary (s.s.s.) if its probabilistic description is invariant to shifts in the time origin.

$$P_{X_{1} \ldots X_{n}} = P_{X_{1} \ldots X_{n}}$$

$\forall c, \forall n, \text{ and } \forall x_{1}, \ldots, x_{n}$.

N.b. The first-order pdf and cdf of a sss RP will be independent of time.

$$f_{Z(t)}(x; t) = f_{Z(t+c)}(x; t+c) = f_{Z(t)}(x), \forall c.$$ 

- The second-order cdf or pdf will be a fn of only the time difference, $T_{12} = t_{1} - t_{2}$

$$f_{Z(t_{1})Z(t_{2})} = f_{Z(t_{1}+c)Z(t_{2}+c)}$$
**Wide-sense stationary (WSS) RP**

A RP \( Z(t) \) is called WSS if it satisfies the following conditions:

1. \( \mathbb{E}[Z(t)] = \eta_x \) (constant)
2. \( \mathbb{E}[Z(t_1) Z^*(t_2)] = R_{xx}(t_1, t_2) = R_x(\tau), \quad \tau = t_1 - t_2 \)

**Note:**

\[ SSS \Rightarrow WSS \]

\[ WSS \nRightarrow SSS \text{ (w/ Gaussian RP is an exception)} \]

**System of Stochastic Process**

\[ Z(t) \rightarrow [\mathbb{T}] \rightarrow Y(t) \]

\[ Y(t, w) = \mathbb{T}[Z(t, w)], \quad \forall w \in \mathcal{X} \]

of particular interest are

1. **Memoryless system.** \( Y(t) = g(Z(t)) \)
   Not a function of time or past \( Z \)

   \[ SSS \rightarrow [\mathbb{\square}] \rightarrow SSS \]

2. **LTI system.**

   \[ WSS \rightarrow [\mathbb{\square}] \rightarrow WSS. \]
The Poisson Point Process.

**Defn:** A *point process* is a set of random points on the time axis:

**1.** The "points" represent the times at which some random events in time occur.

**2.** The $t_i$ are RVs defined on some random experiment $(S, F, P)$. They are "ordered" $t_1 \leq t_2 \leq \cdots < t_n \leq \cdots$.

**Defn:** To each point process $\{t_i\}$ we can associate a *renewal process*, a sequence of RVs $\{Z_n\}$ defined by

$$Z_n = \begin{cases} t_1, & n=1 \\ t_n - t_{n-1}, & n=2, 3, \ldots \end{cases}$$

The renewal process gives the time between "events" in the point process.
Relationship between point process \{t_i\}, renewal process \{Z_i\}, and counting process \{X(t)\}.

**Defn:** A point process \{t_i\} is called a set of Poisson Points if

1. The number of points \(N(t_1, t_2)\) in the interval \([t_1, t_2]\) is a Poisson RV with parameter (mean) \(\lambda(t_2 - t_1)\)
   \[P(\{N(t_1, t_2) = k\} = \frac{e^{-\lambda(t_2-t_1)}}{k!} \left[\lambda(t_2-t_1)\right]^k, \lambda > 0\]

2. If \([t_1, t_2) \cap [t_3, t_4) = \emptyset\), then \(N(t_1, t_2)\) and \(N(t_3, t_4)\) are stat. independent. (for \(t_1 < t_2 < t_3 < t_4\))

**Defn:** The RP \(X(t) = N(0, t)\) corresponding to a set of Poisson points is called Poisson counting process

\[E[X(t)] = \lambda t\]
\[R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] = \lambda \min(t_1, t_2) + \lambda^2 t_1 t_2\]