§ Probability Space

- Random experiment \( \rightarrow \) Random outcome
- Characterized by a probability (\( \mathcal{S}, \mathcal{F}(\mathcal{S}), P \)) made up of 3 elements.
  1. Sample space \( \mathcal{S} \): possible outcome \( w \in \mathcal{S} \).
  2. Possible events described are subsets of \( \mathcal{S} (\mathcal{F}(\mathcal{S})) \).
  3. The probability of \( A \in \mathcal{F}(\mathcal{S}) \) is \( P(A) \).

\( \emptyset \) Sample Space \( \mathcal{S} \)

Intuitively: A listing of all conceivable finest-grain distinguishable outcomes of a random experiment.

Mathematically: An abstract space or universal set.

n.b. \( \mathcal{S} \) is a non-empty set of elements called outcome. One and only one outcome occurs when we perform a random experiment.
Event Space $\mathcal{F}(\Omega)$

Intuitively: A collection of events (subsets of $\Omega$) which we think of as "physical" events we want to know the probability of.

Mathematically: $\mathcal{F}(\Omega)$ is a family of subsets of $\Omega$ that is closed under certain set operations.

$\mathcal{F}(\Omega)$ should have the following 3 closure properties.

1. If $A \in \mathcal{F}(\Omega)$, then $\overline{A} \in \mathcal{F}(\Omega)$
2. If for some finite $n$, $A_i \in \mathcal{F}(\Omega)$, $i = 1, 2, \ldots, n$
   then $\bigcup_{i=1}^{n} A_i \in \mathcal{F}(\Omega)$ ← finite union
3. If $A_i \in \mathcal{F}(\Omega)$, $i = 1, 2, 3, \ldots$
   then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}(\Omega)$ ← countable union

n.b. It follows from these properties that $\Omega \in \mathcal{F}(\Omega)$ and $\emptyset \in \mathcal{F}(\Omega)$

proof: Suppose $A \in \mathcal{F}(\Omega)$, then $\overline{A} \in \mathcal{F}(\Omega)$.

Further $A \cup \overline{A} = \Omega \in \mathcal{F}$, and $\emptyset = \emptyset = \mathcal{F}(\Omega)$
n.b. What about intersection?

Suppose $A, B \in F$. Is $A \cap B \in F(S)$?

\[
A \cap B = \overline{A \cup B} = \overline{A} \cup \overline{B} \in F
\]

\[
\bigcap_{i=1}^{n} A_i = \overline{\bigcup_{i=1}^{n} A_i} = \overline{\bigcup_{i=1}^{n} A_i} \in F
\]

n.b. A family of sets satisfying these closure properties is called a $\sigma$-field.

n.b. Given any set $S$, the family of all subsets of $S$ is a $\sigma$-field. This $F$ is called the power set $P(S)$ of $S$ is $2^S$. This is the largest possible event space. (Can be too big)

n.b. We choose event space $F$ to be the smallest $\sigma$-field containing the desired subsets.

n.b. Borel Field $B(\mathbb{R})$ is a $\sigma$-field containing all open intervals in $\mathbb{R}$. 
Probability Measure $P(\cdot)$

Intuitively: Assign a number between 0 and 1 that measures the certainty or likelihood with which an event will occur.

Mathematically: A set function $P : \mathcal{F}(\Omega) \rightarrow \mathbb{R}$ satisfying the axioms of probability

Axioms of Probability

1. $P(A) \geq 0$, $\forall A \in \mathcal{F}$
2. $P(\Omega) = 1$.
3. If $A_1, A_2 \in \mathcal{F}$ and $A_1$ and $A_2$ are disjoint,
\[ P(A_1 \cup A_2) = P(A_1) + P(A_2) \]
(n.b. If followed by induction that if $A_1, \ldots, A_n \in \mathcal{F}$ are disjoint,
\[ P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) \])
4. If $A_1, A_2, \ldots, A_n \in \mathcal{F}$ are disjoint,
\[ P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i) \]
n.b. The Probability measure $P(.)$ is defined on $\mathcal{F}(\Omega)$ not on $\Omega$.

If you want to talk about the probability of a particular outcome $\omega \in \Omega$, you do so by considering the prob. of $\{\omega\} \in \mathcal{F}(\Omega)$.

$P(\{\omega\})$ is well defined

$P(\omega)$ is not

Remember: $P : \mathcal{F}(\Omega) \to \mathbb{R}$

$P : \Omega \to \mathbb{R}$

Example of $(\Omega, \mathcal{F}, P)$

$\Omega = \{0, 1\}$

$\mathcal{F} = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$

$P(A) = \begin{cases} \frac{1}{4}, & A = \{0\} \\ \frac{3}{4}, & A = \{1\} \\ 0, & A = \emptyset \\ 1, & A = \Omega \end{cases}$

$P(.)$ satisfies the axioms of prob. and is hence a valid prob. measure.
Immediate Properties of \( P(\cdot) \):

(a) \( P(\overline{A}) = 1 - P(A), \ \forall A \in \mathcal{F} \)

**Proof:** \( A \cup \overline{A} = \Omega \Rightarrow P(A \cup \overline{A}) = 1 \) \( (A \times 2) \)
\[ A \cap \overline{A} = \emptyset, \Rightarrow 1 = P(A \cup \overline{A}) = P(A) + P(\overline{A}) \]

(b) \( P(A) \leq 1, \ \forall A \in \mathcal{F} \)

**Proof:** \( P(A) = 1 - P(\overline{A}) \leq 1 \)
\[ \text{Since } P(\overline{A}) \geq 0 \text{ by axiom 1.} \]

(c) \( P(\emptyset) = 0 \)

**Proof:** \( \emptyset = \overline{\Omega}, \text{ so from (a) above, } P(\emptyset) = 1 - P(\Omega) = 0 \)

(d) If \( \{A_i, i \in I\} \) is a finite or countable partition of \( \Omega \), then
\[ P(G) = \sum_{i \in I} P(G \cap A_i), \ \forall G \in \mathcal{F} \]

**Proof:** \( P(G) = P(G \cap \Omega) = P(G \cap (\bigcup_{i \in I} A_i)) \)
\[ = P(\bigcup_{i \in I} (G \cap A_i)) = \sum_{i \in I} P(G \cap A_i) \]
Example of Probability Spaces

**Ex 1** Let $\mathcal{S}$ be finite and $\mathcal{F}(\mathcal{S})$ be the power set of $\mathcal{S}$. Suppose we have a function $\rho(w): \mathcal{S} \rightarrow \mathbb{R}$ s.t.

$$\begin{cases}
1. \rho(w) \geq 0, \forall w \in \mathcal{S}, \text{ and} \\
2. \sum_{w \in \mathcal{S}} \rho(w) = 1
\end{cases}$$

$\rho(w)$ is called a probability mass function (pmf).

Now define prob. meas. $P(\cdot)$

$$P(A) = \sum_{w \in A} \rho(w), \forall A \in \mathcal{F}$$

$$\rho(w) = P(\{w\})$$

$P(\cdot)$ so constructed satisfies axioms of prob.

n.b. You can use the pmf for any discrete space.
**Uniform pmf**

\[ S = \mathbb{Z}_n = \{0, 1, \ldots, n-1\} \]

\[ F = \mathcal{P}(S), \text{ power set of } S \]

\[ |F| = 2^n \]

pmf: \[ p(w) = \frac{1}{n}, \forall w \in S \]

\[ P(\cdot), \text{ probability measure} \]

\[ P(A_k) = \sum_{w \in A_k} p(w), \quad A_1, A_2, \ldots A_{2^n} \]

\[ = \frac{|A_k|}{|S|} = \frac{|A_k|}{n} \]

**Binomial pmf**

\[ S = \{0, 1, 2, \ldots, n\} \]

\[ F = \mathcal{P}(S), \quad |F| = 2^{n+1} \]

pmf: \[ p(k) = \binom{n}{k} a^k (1-a)^{n-k}, \quad a \in [0, 1] \]

\[ P(\cdot): P(A) = \sum_{k \in A} p(k) = \sum_{k \in A} \binom{n}{k} a^k (1-a)^{n-k} \]

n.b. \[ \binom{n}{k} = \frac{n!}{k! (n-k)!} \]

**Exercise** Show \[ P(S) = \sum_{k=0}^{n} p(k) = 1 \]
**Geometric pmf**

\[ S = \{1, 2, \ldots \} \quad \Omega = \mathbb{N} \]

\[ F = \mathcal{P}(S) \]

pmf: \[ p(k) = (1-a)^{k-1} a, \quad a \in [0, 1] \]

\[ P(\cdot) : P(A) = \sum_{k \in A} p(k) = \sum_{k \in A} (1-a)^{k-1} a \]

**Exercise**: Show \( P(\mathbb{N}) = 1 \)

**Poisson pmf**

\[ S = \mathbb{Z}_+ = \{0, 1, 2, \ldots \} \]

\[ F = \mathcal{P}(S) \]

pmf: \[ p(k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k=0, 1, 2, \ldots \quad \lambda > 0 \]

\[ P(\cdot) : P(A) = \sum_{k \in A} p(k), \quad \forall A \in F \]

**Exercise**: Show \( P(\mathbb{N}) = 1 \)
Ex2 Let $(\mathcal{S}, F) = (\mathbb{R}, B(\mathbb{R}))$ and consider a real-valued function $f: \mathbb{R} \to \mathbb{R}$ with props:

1. $f(r) \geq 0$, $\forall r \in \mathcal{S} = \mathbb{R}$
2. $\int_{\mathbb{R}} f(r) \, dr = \int_{-\infty}^{\infty} f(r) \, dr = 1$

We call such a function $f(\cdot)$ a prob. density ftn (pdf)

$P(\cdot)$: For any event $A \in B(\mathbb{R})$

$$P(A) \triangleq \int_{A} f(r) \, dr = \int_{-\infty}^{\infty} f(r) \, 1_{A}(r) \, dr$$

where $1_{A}(r) \triangleq \begin{cases} 1 & r \in A \\ 0 & r \notin A \end{cases}$

is called the indicator function of set $A$.

Q? Does $P(A) = \int_{\mathbb{R}} f(r) \, 1_{A}(r) \, dr$ give a valid prob. measure $P(\cdot)$ for all $A \in B(\mathbb{R})$?
There are a number of common pdf that lead to prob. space \((\mathbb{R}, B(\mathbb{R}), P)\)

**Uniform pdf**

\[ f(r) = \frac{1}{b-a} 1_{[a,b]}(r) , \ a < b \]

**Exponential pdf**

\[ f(r) = \lambda e^{-\lambda r} 1_{[0,\infty)}(r) , \ \lambda > 0 \]

**Gaussian pdf**

\[ f(r) = \frac{1}{\sqrt{2\pi} \sigma} \exp \left\{ \frac{-(r-\mu)^2}{2\sigma^2} \right\} , \ \mu \in \mathbb{R} , \ \sigma > 0 \]
The Calculus of Probability

**Theorem 1.** If $P$ is a prob. measure, and $A$ is any set in $\mathcal{F}$, then:

- a. $P(\emptyset) = 0$
- b. $P(A) \leq 1$
- c. $P(\overline{A}) = 1 - P(A)$

**Theorem 2.** If $P$ is a prob. measure, and $A$ and $B$ are any sets in $\mathcal{F}$:

- a. $P(B \Delta \overline{A}) = P(B) - P(A \cap B)$
- b. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- c. If $A \subseteq B$, then $P(A) \leq P(B)$

Bonferroni's Inequality from (b):

$P(A \cap B) \geq P(A) + P(B) - 1$

**Proof:**

- **(a)** $B = \{B \cap A\} \cup \{B \cap \overline{A}\}$
  \[ P(B) = P(B \cap A) + P(B \cap \overline{A}) \]

- **(b)** $A \cup B = A \cup \{B \cap \overline{A}\}$
  \[ P(A \cup B) = P(A) + P(B \cap \overline{A}) = P(A) + P(B) - P(A \cap B) \]

- **(c)** If $A \subseteq B$, then $A \cap B = A$
  \[ 0 \leq P(B \cap \overline{A}) = P(B) - P(A) \]
Theorem 3. If $P$ is a prob. measure, then

a. $P(A) = \sum_{i=1}^{\infty} P(A \cap G_i)$ for any partition $G_1, G_2, \ldots$

b. $P\left( \bigcup_{i=1}^{\infty} A_i \right) \leq \sum_{i=1}^{\infty} P(A_i)$ for any sets $A_1, A_2, \ldots$

Boole's Inequality