**Point Estimate Method**

\[ Y = g(x) \]

- A simple procedure for computing the first three moments by sacrificing the accuracy of a rigorous probabilistic analysis; the procedure also furnishes an approximate and equally simple approach to Bayesian statistics.

- The method gives results that are usually almost as satisfactory as those of a rigorous probabilistic treatment provided the coefficients of variation of independent variables do not exceed moderate limit, yet it implies no more than a modest increase in numerical complexity over that of a purely deterministic analysis.

**Key:**

Replacing the marginal pdf of the independent variable \( x \) by a discrete pdf of two points, \( P \) at \( x_\cdot \) and \( P_\cdot \) at \( x_\cdot \). \( P_\cdot \) and \( P_\cdot \) are said to be two-point estimates of pdf \( p(x) \).
Distributed load analogy

![Graph showing distributed load analogy](image)

**Figure 4.3.1** Continuously distributed vertical force system on rigid beam. (a) Single support. (b) Two supports.

Function of one variable

We are interested in obtaining expressions for the expectation, standard deviation, and coefficient of skewness, and to do it independent of $x$'s distribution. To this end, let us assign $X$ an arbitrary pdf having four parameters $\alpha$ to comply with expressions for the moment of zero and for the first three moments of $X$. A particular simple function satisfying this requirement consist in two concentration $P$ and $Q$ at $x_-$ and $x_+$.

$$P_x(x) = P \delta(x-x_-) + Q \delta(x-x_+)$$
The four expressions that are of interest become

\[
\begin{align*}
1 &= P_+ + P_- \\
\bar{x} &= P_+ x_+ + P_- x_- \\
\sigma^2 [x] &= P_+ (x_+ - \bar{x})^2 + P_- (x_- - \bar{x})^2 \\
\beta_1 [x] \sigma^3 [x] &= P_+ (x_+ - \bar{x})^3 + P_- (x_- - \bar{x})^3
\end{align*}
\]

or

\[
\begin{align*}
1 &= P_- + P_+ \\
0 &= \xi_1 P_+ - \xi_2 P_- \\
1 &= \xi_3^2 P_- + \xi_2 P_+ \\
\beta_1 [x] &= \xi_3^3 P_- - \xi_3^2 P_+ \\
\text{where } \xi_1 &= \frac{|x_i - \bar{x}|}{\sigma}
\end{align*}
\]

The solution to the above equations are

\[
\begin{align*}
P_+ &= \frac{1}{2} \left[ 1 + \sqrt{1 - \frac{1}{1 + (\beta_1 / \xi_1)^2}} \right] \\
P_- &= 1 - P_+ \\
x_+ &= \bar{x} + \sigma [x] \sqrt{\frac{P_-}{P_+}} \\
x_- &= \bar{x} - \sigma [x] \sqrt{\frac{P_+}{P_-}}
\end{align*}
\]

* If \( \beta_1 \) is unknown or zero

\[
\begin{align*}
P_+ &= \frac{1}{2} \\
P_- &= \frac{1}{2} \\
x_+ &= \bar{x} + \sigma [x] \\
x_- &= \bar{x} - \sigma [x]
\end{align*}
\]
\[ y = g(x) \]

**Figure 4.8.2** Schematic representation of transfer of information.

\[
\begin{align*}
X^+ & \rightarrow y^+ = g(x^+) & P_+ & \rightarrow P^+ \\
X^- & \rightarrow y^- = g(x^-) & P_- & \rightarrow P_-
\end{align*}
\]

\[ E[y^n] = P_- y_-^n + P_+ y_+^n \]

Given the noncentral moments \( m_0, m_1, \ldots, m_n \),
we can compute the central moment \( \mu_0, \ldots, \mu_n \)
\[
\mu_n = E\left\{ (X - \bar{X})^n \right\} = E\left\{ \sum_{k=0}^{n} \binom{n}{k} X^k (-\bar{X})^{n-k} \right\}
\]

**Note:** Point estimate and their points of application, in concept, serve to transfer information about the distribution of the variate

\[ y_i = g(x_i) \quad \text{and} \quad P_i \; \text{scale the estimate} \]
Analysis of error of approximation

Rosenbluth (1981) found that

$Y_i$ : 3rd order approximation ($\sim$ relative error of the order of $V^4$)

$\Delta Y_i$ : 2nd order approximation ($\sim V^3$)

$\beta_i$ : 1st order approximation ($\sim V^2$)

If $\beta_i(x)$ is unknown

$\bar{Y} = 2nd$ order approximation

$\Delta Y_i$ : 1st order approximation

Bayesian Statistics

$$P(m | \text{dobs}) = \frac{P(\text{dobs} | m) p(m)}{\int P(\text{dobs} | m) p(m) \, dm}$$

$$P(m = m_i | \text{dobs}) = \frac{P(\text{dobs} | m = m_i) p(m_i)}{\sum P(\text{dobs} | m = m_i) p(m_i)} \quad \text{(*)}$$

Points remains the same at $m_i$, but

$$P(m_i) \xrightarrow{\text{transform through (*)}} P(m = m_i | \text{dobs})$$

Note: In the uncertainty analysis $Y = g(x)$

$$P(y_i) = P(x_i), \text{ but } x_i \rightarrow y_i = g(x_i)$$
Function of several variables

Generalization of the foregoing method to functions of several variables requires solution of large numbers of simultaneous equations, many of which are generally non-linear.

Let $n$ be the number of R.V.'s, and $N_i$ the number of moments of order $i$ to be satisfied. Then the number of conditions to be met, and hence the number of imposed equis, for all moments, from the one of order 0 to those of order $k$,

$$\sum_{i=0}^{k} N_i = \frac{(k+n)!}{k! \cdot n!}$$

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
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<tbody>
<tr>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$N_n + N_{n-1}$</td>
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<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>$N_n + N_{n-1} + N_{n-2}$</td>
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<td>6</td>
<td>10</td>
<td>15</td>
<td>21</td>
<td>28</td>
<td>36</td>
</tr>
<tr>
<td>$N_n + N_{n-1} + N_{n-2} + N_{n-3}$</td>
<td>4</td>
<td>10</td>
<td>20</td>
<td>35</td>
<td>56</td>
<td>84</td>
<td>120</td>
</tr>
</tbody>
</table>

For every point where the probability density function $p$ is assumed concentrated, we can write as many equations as the point has coordinates, that is $n$, plus one, the latter coming from the magnitude of the concentration. The smallest number of concentration is then

$$\frac{1}{n+1} \sum_{i=0}^{K} N_i$$
Table A.2  Number of concentrations and number of redundant parameters

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
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</tr>
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<td>10</td>
<td>4</td>
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<td>0</td>
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<tr>
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<td>15</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

(1) $n$
(2) $N_0 + N_1 + N_2 + N_3$ = number of equations imposed by number of moments of $p$
(3) $(N_0 + N_1 + N_2 + N_3)/(n+1)$ = lower bound to number of points where $p$ is to be concentrated
(4) smallest number of concentrations of $p$
(5) number of redundant parameters
(6) ditto when using $2^n$ concentrations of $p$

Note: (6): $2^n(n+1) - \sum_{i=0}^{k} N_i$

This approach is awkward. We prefer to concentrate the density function at a superabundant number of points and impose conditions on their coordinates. If we take $2^n$ points when the number of random variable is $n$ and we leave as unknowns the concentrations at all points and coordinates of two of them not having coordinates in common, distributing the rest so as to form a rectangle, prism, or hyper prism.

$\Rightarrow$ number of unknown $N_u = 2^n + 2n$.

The proposed solution matches up to second-order crossed and third-order non crossed moments of $X_1, X_2, \ldots, X_n$.

$\Rightarrow$ number of equation $N_e = 1 + 3n + n(n-1)/2$
Table 1. Comparison of number of equations and number of unknowns.

<table>
<thead>
<tr>
<th>n</th>
<th>$N_s$</th>
<th>$N_u$</th>
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<td>9</td>
<td>64</td>
<td>530</td>
</tr>
<tr>
<td>10</td>
<td>76</td>
<td>1,044</td>
</tr>
</tbody>
</table>

For $n = 2$, and $\beta_1(x_1) = \beta_1(x_2) = 0$.

Figure 4.3.1 Representation of two variables as a loading on rigid plate.

Figure 4.3.2 Influence of correlation coefficient on weighting factors.
For $n=3$, $y = f(x_1, x_2, x_3)$, and $\beta_1(x_i) = \beta_1(x_3) = \beta_1(x_3) = 0$

\[
\begin{align*}
Y_{\pm\pm} &= Y(\bar{x}_1 \pm \delta(x_1), \bar{x}_2 \pm \delta(x_2), \bar{x}_3 \pm \delta(x_3)) \\
P_{++} &= P_{--} = \frac{1}{2^3} (1 + P_{12} + P_{23} + P_{31}) \\
P_{+-} &= P_{-+} = \frac{1}{2^3} (1 + P_{12} - P_{23} - P_{31}) \\
P_{-+} &= P_{++} = \frac{1}{2^3} (1 - P_{12} - P_{23} + P_{31}) \\
P_{+-} &= P_{-+} = \frac{1}{2^3} (1 - P_{12} + P_{23} - P_{31})
\end{align*}
\]
Rosenbluth's Algorithm (2ⁿ method)

In summary, for one random variable

\[
\begin{align*}
X_+ &= \mu + Z_+ \sigma \\
X_- &= \mu - Z_- \sigma \\
P_+ &= \frac{Z_-}{Z_+ + Z_-} \\
P_- &= 1 - P_+ \\
\end{align*}
\]

where

\[
Z_+ = \frac{\beta}{\sigma} + \sqrt{1 + \left(\frac{\beta}{\sigma}\right)^2}
\]

\[
Z_- = Z_+ - \beta
\]

\[
\mu: \text{ mean}
\]

\[
\sigma: \text{ standard deviation}
\]

\[
\beta: \text{ coefficient of skewness}
\]

For \( Y = f(x) = g(x_1, x_2, \ldots, x_n). \) The points in the parameter space are determined by permuting the two points in each dimension resulting in \( 2^n \) points.

\[
E[Y^m] = \sum_{\delta_1 = \pm} \ldots \sum_{\delta_n = \pm} P(\delta_1, \ldots, \delta_n) y^m(\delta_1, \ldots, \delta_n)
\]

where

\[
P(\delta_1, \ldots, \delta_n) = \prod_{i=1}^{n} P_i \delta_i + \left( \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_i \delta_j a_{ij}\right)
\]

\[
a_{ij} = \frac{E_{ij}/2^n}{\sqrt{\prod_{i=1}^{n} \left(1 + \left(\frac{\delta_i}{\sigma_i}\right)^2\right)}}
\]

where \( P_i \delta_i \) represents the probability mass at point location \( X_i \delta_i \) (i.e. \( X_i+ \) or \( X_i- \)).
When \( n \) is large, besides the inefficiency of the \( 2^n \) point estimate method, it is time consuming to use because to calculate the moments of a function of \( n \) variables is \( 2^n \). For example, when \( n=10 \) and \( 20 \), one needs evaluate \( 1024 \) and \( 1,048,576 \) times.

Alternative procedures that use \( 2n \) point concentrations match up to second-order moment.

Mathematical Background

- Linear Transformation

\[
X = \begin{bmatrix}
X_1 \\
X_2 \\
\vdots \\
X_n
\end{bmatrix}, \\
M_x = \begin{bmatrix}
\mu_1 \\
\mu_2 \\
\vdots \\
\mu_n
\end{bmatrix} = E[X], \\
\Sigma_x = E \left\{ (X - M_x)(X - M_x)^T \right\}
\]

for the linear transformation

\[
W = AX \\
\begin{bmatrix}
W_1 \\
W_2 \\
\vdots \\
W_n
\end{bmatrix} = \begin{bmatrix}
a_1 & \cdots & a_m
\end{bmatrix} \begin{bmatrix}
X_1 \\
X_2 \\
\vdots \\
X_n
\end{bmatrix}
\]

\[
\Rightarrow M_w = E[W] = A^T E[X] = A^T M_x \\
\Sigma_w = E \left\{ (W - M_w)(W - M_w)^T \right\} = E \left\{ A^T(X - M_x)(X - M_x)^T A \right\} \\
= A^T E \left\{ (X - M_x)(X - M_x)^T \right\} A = A^T \Sigma_x A
\]

\[
\downarrow \begin{aligned}
M_w &= A^T M_x \\
\Sigma_w &= A^T \Sigma_x A
\end{aligned}
\]
Principal Component Transformation

Let

$$\Sigma_x \phi_i = \lambda_i \phi_i, \ i = 1, 2, \ldots, n.$$  

\(\lambda_i\) : eigenvalue of \(\Sigma_x\), \(\phi_i\) : eigenvector.

\[\therefore \Sigma_x = \Sigma_x^T\]

\[\therefore \lambda_i, \phi_i\] are real and \(\phi_i^T \phi_j = 0, i \neq j\)

\[\phi_i^T \phi_i = \| \phi_i \|^2\]

\[\Sigma_x [\phi_1, \ldots, \phi_n] = [\lambda \phi_1, \ldots, \lambda_n \phi_n] = [\phi_1, \ldots, \phi_n] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \ddots & \ddots \\ 0 & \cdots & \lambda_n \end{bmatrix} = \Phi_{mxn} \Lambda\]

\[\Rightarrow \Sigma_x \Xi = \Xi \Lambda \text{ and } \Xi^T \Xi = I_{mxn}, \Xi^T = \Xi^T\]

\[\Rightarrow \Lambda = \Xi^T \Sigma_x \Xi = \Xi^T \Sigma_x \Xi\]

So,

\[\text{If } W = \Xi^T X\]

\[\Rightarrow \Sigma_W = \Phi^T \Sigma_x \Phi = \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \ddots & \ddots \\ 0 & \cdots & \lambda_n \end{bmatrix}\]

\[X \xrightarrow{\Sigma_x} W = \Xi^T X \xrightarrow{W} Z = \Lambda^{\frac{1}{2}} W \xrightarrow{Z} I\]

\[\Sigma_Z = \Lambda^{\frac{1}{2}} \Lambda \Lambda^{\frac{1}{2}} = I\]

\[\Lambda^{\frac{1}{2}} = \begin{bmatrix} \sqrt{\lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{\lambda_n} \end{bmatrix}\]

*Whitening Process!*
Principal Component obtained from standardized random variable

Principal Component may also be obtained for standardized random variables.

\[ H_1 = \frac{(X_1 - \mu_1)}{\sqrt{\sigma_1}} \]
\[ H_2 = \frac{(X_2 - \mu_2)}{\sqrt{\sigma_2}} \]
\[ H_n = \frac{(X_n - \mu_n)}{\sqrt{\sigma_n}} \]

In matrix notation

\[ H = \sqrt{\frac{1}{n}} (X - M_x) \]

where \( V = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n \end{bmatrix} \)

\[ \Rightarrow \quad M_H = 0 \]

\[ \Sigma_H = \sqrt{\frac{1}{n}} \Sigma_x \cdot \sqrt{\frac{1}{n}} = \Sigma_x \]

\[ \Sigma_x \Phi_i = \lambda_i \Phi_i \]

\[ X \rightarrow H = \sqrt{\frac{1}{n}} X \]
\[ \Phi_x \rightarrow W = \Phi^T H \]
\[ \Lambda \rightarrow Z = \Lambda^{\frac{1}{2}} W \]
\[ \Lambda \rightarrow I \]
n.b.

$\Sigma$ and $\Lambda$ obtained from $\Sigma_x$ are different from that obtained from $\Sigma_x$. Furthermore one set of principal component ($\lambda_i$'s) is not a simple function of the other. This suggests that the standardization is not inconsequential.

Example.

Consider the covariance matrix $\Sigma = \begin{bmatrix} 1 & 4 \\ 4 & 100 \end{bmatrix}$ and the derived correlation matrix $\Phi = \begin{bmatrix} 1 & 0.4 \\ 0.4 & 1 \end{bmatrix}$

\[
\Sigma : \Lambda = \begin{bmatrix} 100.1614 & 0 \\ 0 & 0.8386 \end{bmatrix}, \quad \Phi = \begin{bmatrix} 0.9992 & 0.0403 \\ 0.0403 & -0.9992 \end{bmatrix}
\]

\[
\Phi : \Lambda = \begin{bmatrix} 1.4 & 0 \\ 0 & 0.6 \end{bmatrix}, \quad \Phi = \begin{bmatrix} 0.7071 & 0.7071 \\ 0.7071 & -0.7071 \end{bmatrix}
\]

\[
\Sigma: \frac{\lambda_1}{\lambda_1 + \lambda_2} = \frac{100.16}{101} = 0.992
\]

\[
\Phi: \frac{\lambda_1}{\lambda_1 + \lambda_2} = \frac{1.4}{2} = 0.7
\]
Transformation of RVs

(a) sample space  

(b) transformed  

(c) rotated  

(d) standard sample space  

Figure 1. Scatter plots of a set of random variables.

(b) Translation  

(c) Rotation by principal component transformation  

(d) Standardization to variance = 1, by dividing by $a_i$

Note that the rotation of random variable is based on the covariance matrix $(\Sigma)$ of random vector $\Xi$. It is possible mathematically to rotate the variables $\Xi$ based on correlation matrix, which are dimensionless. However, a distinct advantage of using a covariance matrix is that the statistical feature is kept unchanged so that the characteristic points, such as A, B, C, and D, can be obtained easily.
Rotation based on correlation matrix is equivalent to standardizing the variable before the rotation. The scatter plot is then distorted to a certain degree.

\[ d_s = \sqrt{y_1^2 + y_2^2 + \ldots + y_n^2} \]

It can be easily seen that a locus of constant statistical distance is a hypersphere. The probability density is the same on the locus. In particular, when \( y_1, y_2, \ldots, y_n \) are equal to ±1, the statistical distance \( d_s \) is equal to \( \sqrt{n} \). This statistical distance takes on special significance, as the Rosenblueth 2^n points are laid on the locus.
Alternative PEM Methods

(1) Principal Component Transformation using Covariance Matrix

(a) \[ X \xrightarrow{\text{Translation}} X' \xrightarrow{\Sigma_X} W = \Xi^T X' \]

\( W \) becomes uncorrelated but the standard deviations are not standardized.

Select 2 points on each \( Wi \) axis, there are total of 2n terms. The locations are

\[
\begin{align*}
W_{i-} &= \{ 0, 0, \ldots, -\sqrt{n}, \ldots, 0 \} \\
W_{i+} &= \{ 0, 0, \ldots, +\sqrt{n}, \ldots, 0 \}
\end{align*}
\]

The weights are

\[
P_{i-} = P_{i+} = \{ 0, 0, \ldots, \frac{\lambda_i}{\sum \lambda_i}, \ldots, 0 \}
\]

The points selected on \( W \) space can be transformed back to \( X \) space by

\[
X = M_X + \Xi W
\]
It can be easily seen that a locus of constant statistical distance is a hypersphere. The probability density is the same on the locus. In particular, when $Z_i$'s are equal to $\pm 1$, the statistical distance is equal to $\sqrt{n}$. This statistical distance takes on special significance, as the Rosenbluth $2^n$ points are laid on the locus.

To obtain an unbiased estimate, the points should be arranged so that they are symmetrical with axes of all variables.

(i) $2n$ points: which are intersections of the variable axes and the hypersphere, provide the most effective PEM calculation scheme.

(ii) $2^n$ points: one the lines $\pm Z_1, \pm Z_2 = \cdots = \pm Z_n = \pm 1$ provide another good alternative.

(iii) $2n + 2^n$ points provide a third alternative.
For $2n$ terms:

$$Z_{i-} = \{0, 0, \ldots, -\sqrt{n}, \ldots, 0\}$$
$$Z_{i+} = \{0, 0, \ldots, +\sqrt{n}, \ldots, 0\}$$

For $2^n$ terms

$$Z_{\pm \ldots \pm} = \{\pm 1, \pm 1, \ldots, \pm 1\}$$

The weight for each point is

$$\rho = \frac{1}{q}, \text{ where } q = 2n, 2^n, \text{ or } 2n + 2^n$$

The points selected on $Z$ space can be transformed back to $X$ space by

$$X = M_x + \Phi A \frac{1}{\rho} Z$$
(2) Principal Component Transformation using Correlation Matrix.

Parallel to the development of covariance matrix.

(c) \[
\begin{align*}
X &\xrightarrow{\text{Translation}} X' \\
&\xrightarrow{\Sigma_x} \text{Standardization} \\
&\xrightarrow{H = \sqrt{\frac{1}{2}} X'} \text{rotation} \\
&\xrightarrow{W = \Phi^T H} W
\end{align*}
\]

Select 2n points on each \(W_i\) axis:

- \(W_i^- = \{0, 0, \ldots, -\sqrt{n}, \ldots, 0\}\)
- \(W_i^+ = \{0, 0, \ldots, \sqrt{n}, \ldots, 0\}\)

The weights are

- \(P_i^- = P_i^+ = \{0, 0, \ldots, \frac{\lambda_i}{\Sigma \lambda_i}, \ldots, 0\}\)

The points selected on \(W\) space can be transformed back to \(X\) space by

\[
X = M_x + V^{\frac{1}{2}} \Phi W
\]
We can pick:

(i) $2n$ terms
(ii) $2^n$ terms
(iii) $2n + 2^n$ terms.

For $2n$ terms:

$Z_{i-} = \{0, 0, \ldots, -\sqrt{n}, \ldots, 0\}$

$Z_{i+} = \{0, 0, \ldots, +\sqrt{n}, \ldots, 0\}$

For $2^n$ terms:

$Z_{\pm \ldots \pm} = \{\pm 1, \pm 1, \ldots, \pm 1\}$

The weight for each point is:

$p = \frac{1}{q}$, where $q = 2n, 2^n$ or $2n + 2^n$.

The points selected on $Z$ space can be transformed back to $X$ space by

$X = Mx + V^{\frac{1}{2}} \Phi \Lambda^{\frac{1}{2}} Z$