EFFICIENT RELIABILITY EVALUATION USING SPREADSHEET

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ABSTRACT: A practical method using spreadsheets is proposed for calculating the Hasofer-Lind second moment reliability index $\beta$. Two example problems involving correlated normals and correlated nonnormals, respectively, are solved and compared with established mathematical approach. The proposed method is based on the perspective of an ellipsoid that is tangent to the failure surface in the original space of the random variables; concepts of transformed space or reduced variables are not required. Correlation is accounted for by setting up the quadratic form in the spreadsheet. Nonnormals are dealt with using established relationships between nonnormal distribution and its equivalent normal distribution. Iterative searching is performed automatically by invoking a spreadsheet’s optimization tool. An advantage of the spreadsheet method is that partial derivatives are not part of the input; the method may therefore be especially attractive for cases with complicated or nonexplicit performance functions.

INTRODUCTION

The second moment reliability index $\beta$ as defined by Hasofer and Lind (1974) is generally considered to be a better approach in civil engineering design than the conventional factor of safety concept. The index reflects not only the effect of the mean values but also the covariances of the random variables influencing the design. The matrix formulation of the Hasofer-Lind index was given, for example, in Ditlevsen (1981), citing Veneziano (1974)

$$\beta_{ML} = \min_{x \in F} \sqrt{(x - m)^T C^{-1} (x - m)}$$

where $x$ is a vector representing the set of random variables; $m$ = their mean values; $C$ = covariance matrix; and $F$ = failure region. A widely used procedure for computing the $\beta$ index by (1) is to transform the failure surface into the space of reduced variables, whereby the shortest distance from the transformed failure surface to the origin of the reduced variables is the reliability index $\beta$. The computation procedure is well explained in Ditlevsen (1981), Ang and Tang (1984), Madsen et al. (1986), Melchers (1987), Dai and Wang (1992), and Tichy (1993), among others.

A more intuitive interpretation of $\beta$ is possible by noting that (1) literally suggests that the Hasofer-Lind index can be calculated by minimizing the quadratic form (in this case an ellipsoid) subject to the constraint that the ellipsoid just touches the surface of the failure region $F$. The convenience and clarity of this perspective for the proposed spreadsheet method of calculating the reliability index are shown in this paper. Cases involving single performance function are considered here. The concept and procedure could be extended to systems with multiple performance functions.

ALTERNATIVE PERSPECTIVE OF HASOFER-LIND INDEX

In the original space of the random variables, one can define a one-standard-deviation ($1-\sigma$) dispersion ellipsoid by its canonical form if the variables are uncorrelated; each axis of the ellipsoid is parallel to a corresponding coordinate axis. When there is dispersion the correlation the ellipsoid is tilted. The equation of the tilted ellipsoid is given by the quadratic form in (1), namely

$$(x - m)^T C^{-1} (x - m) = 1$$

This quadratic form is an ellipse for two-dimensional (2D) case and a hyperellipsoid for dimensions greater than three. Eq. (2) is plotted in Fig. 1 for a 2D case with mean values $m_1 = m_2 = 9$, and standard deviations $\sigma_1 = 3$, and $\sigma_2 = 2$, for different values of correlation coefficient $\rho$. When $x_1$ and $x_2$ are uncorrelated, (2) reduces to the familiar expression of an ellipse, the shape of which is labeled $\rho = 0$ in Fig. 1. When $x_1$ and $x_2$ are correlated, the ellipse rotates and changes its aspect ratio. Despite the rotation, the dispersion of the $1-\sigma$ ellipse in the original $x_1$-$x_2$ reference frame is still defined by $\sigma_1$ and $\sigma_2$.

The failure surface shown in Fig. 1 is a second degree polynomial, $x_2 = 25.5 - 1.41x_1 + 0.039x_1^2$, and, for the case with correlation coefficient $\rho = 0.7$, a reliability index $\beta$ equal to 2.18 is obtained using the spreadsheet method described in Low (1996), which will be explained and extended to correlated nonnormals in the next section. The $1-\sigma$ ellipse, the $\beta$-$\sigma$ ellipse, and the failure surface are plotted together in Fig. 1.

FIG. 1. 1-\sigma Dispersion Ellipse Rotates as Correlation Coefficient $\rho$ Changes
2. The equation for the $\beta$-$\sigma$ ellipse is (2), but with the right hand side (RHS) replaced by $\beta^2$, where $\beta = 2.18$ as calculated. In Fig. 2, the ellipse that is tangent to the failure surface is $\beta$ times the size (in terms of axis ratio) of the 1-$\sigma$ dispersion ellipse. This provides an intuitive meaning of the reliability index $\beta$ in the original space of the random variables.

The ellipsoid approach via spreadsheet is intuitively simple and transparent. In solving the constrained nonlinear optimization problem of (1) the user literally asks for the smallest ellipsoid that touches the failure surface; concepts of transformed space and reduced variates [often used in connection with (1)] are not necessary.

There is a connection between the multivariate normal density function and the reliability index $\beta$. In the 2D case the bivariate normal density function is

$$f_{r,x_i}(x_1, x_2) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1 - \rho^2}} \exp \left( -\frac{(x_1 - \mu_1)^2}{2\sigma_1^2} - \frac{(x_2 - \mu_2)^2}{2\sigma_2^2} + \frac{x_1 x_2}{\sigma_1 \sigma_2} \rho \right)$$

(3)

where $\beta$ is as defined in (1), without minimizing.

Therefore, to minimize $\beta$ (or $\beta^2$) is to maximize the value of the bivariate density function. The 1-$\sigma$ dispersion ellipse and the $\beta$-$\sigma$ ellipse of Fig. 2 are contours of probability density function. To find the smallest ellipse (or hyperellipsoid, for multivariate case) that is tangent to the failure surface is then equivalent to finding the most probable failure point. This perspective is consistent with that of Shinozuka (1983) who stated that "the design point $x^*$ is the point of maximum likelihood if $x$ is Gaussian, whether or not its components are uncorrelated."

The proposed spreadsheet method is illustrated in the following two examples, for correlated normals, and correlated nonnormals, and compared with established mathematical procedures. Although Microsoft Excel (version 5 or 7) was used, it is likely that other spreadsheet softwares are (or will be) equally adequate for the tasks in hand.

EXAMPLE 1: CORRELATED NORMALS

Ang and Tang ([1984], Example 6.9) analyzed the reliability of a steel beam section using an established mathematical approach. The performance function was

$$g(X) = Y - M$$

(4)

where $Y$ = yield strength of steel; $Z$ = section modulus; and $M$ = applied bending moment at the pertinent section. The mean values and standard deviations of $Y$, $Z$, and $M$ are known. Also, the variables $Y$ and $Z$ are assumed to be partially correlated, with correlation coefficient $\rho_{YZ} = 0.4$.

In the proposed spreadsheet method, the mean values and the covariance matrix are tabulated as shown in the spreadsheet in Fig. 3. The diagonal terms of the covariance matrix are $\sigma_Y^2$, $\sigma_Z^2$, and $\sigma_M^2$, respectively. The nonzero off-diagonal terms represent $\rho_{YZ}\sigma_Y\sigma_Z$. Subsequent steps are

1. Formulas are entered for the column vector $[x-m]$ and the row vector $[x-m]^T$, where $x$ represents xvalues, and $m$ represents mean values. The xvalues are set equal to the mean values initially.

2. The inverse of the covariance matrix is obtained using the spreadsheet's built-in MINVERSE function: a. Select any blank 3 $\times$ 3 cells; b. Type "=minverse(array)," where "array" is entered by selecting the 3 $\times$ 3 covariance matrix; and c. Press "Enter" while holding down "Ctrl" and "Shift."

3. Similarly, the spreadsheet's MMULT function is used to obtain the matrix product $[C][x-m]$, then $[x-m][C]^T[x-m]$.

4. The formulas of the reliability index, $\beta = \sqrt{\text{det}([x-m][C][x-m])}$, of the performance function, $g(x) = (YZ - M)$, are entered based on xvalues.

5. Solver is invoked, to Minimize $\beta$. By Changing xvalues, Subject To $g(x) = 0$.

The solution obtained by Excel's Solver in step 5 is as shown in Fig. 3. The $\beta$ value of 2.863 is virtually identical to that (2.862) in Ang and Tang (1984). The spreadsheet approach is simpler and more intuitive because it does not involve eigenvalues and eigenvectors, orthogonal transformation matrix, reduced variates, or explicit partial derivatives.

EXAMPLE 2: CORRELATED LOGNORMALS AND TYPE I ASYMPOTIC EXTREME

A case involving correlated nonnormals was considered in Ang and Tang ([1984], Example 6.10). The performance function is the same as (4), except that the variables $Y$ and $Z$ are lognormals, with correlation coefficient $\rho_{YZ}$ equal to 0.4. The
variable \( M \) is Type I extreme. The mean and standard deviations of \( Y, Z, \) and \( M \) are otherwise the same as those in Fig. 3.

The spreadsheet solution for this case (Fig. 4) has the matrix setup of the quadratic form (as in Fig. 3) to account for correlation, and the calculations of the lognormal parameters \( \lambda \) and \( \zeta \) from the mean value and the coefficient of variation \( \Omega \) [e.g., Ang and Tang (1975)]

\[
\lambda = \ln(\text{mean}) - 0.5 \zeta^2; \quad \zeta = \sqrt{1 + \Omega^2} \tag{5a,b}
\]

For any trial \( x \)-value of a nonnormal variate, equivalent normal parameters \( (\mu^*, \sigma^*) \) can be obtained using the established procedure of equating the cumulative probability and the probability density ordinate of the equivalent normal distribution with those of the corresponding nonnormal distribution at the particular \( x \)-value of the random variable. For lognormals the following are obtained [e.g., Ang and Tang (1984)]

\[
m^* = x\text{value} \times (1 - \ln(x\text{value}) + \lambda); \quad \sigma^* = x\text{value} \times \zeta \tag{6a,b}
\]

For variable \( M \), the Type-I-extreme parameters are \( \alpha \) and \( u \), given by

\[
\alpha = \frac{\pi}{\sqrt{6} \mu} = 0.006413; \quad u = \mu_u - \frac{0.5772}{\alpha} = 910 \tag{7a,b}
\]

in which the standard deviation of \( M(\sigma_u) \) is 200 and the mean of \( M(\mu_u) \) is 1,000, as in Fig. 3.

The mean \( (\mu^*) \) and standard deviation \( (\sigma^*) \) for the equivalent normal distribution of \( M \) are obtained from the following equations [e.g., Ang and Tang (1984) Eqs. 6.25 and 6.26]

\[
m^* = M^* - \sigma^* \Phi^-1(F(M*)); \quad \sigma^* = \frac{\Phi^{-1}(F(M*))}{f(M*)} \tag{8a,b}
\]

where \( M^* \) is in this case the value of \( M \) of a point on the failure surface \( g(Y, Z, M) = 0 \); \( \Phi^{-1} \) is inverse of the cumulative probability (CDF) of a standard normal distribution; \( F(M^*) \) = original CDF evaluated at \( M^* \); \( \Phi^{-1} \) = probability density function (PDF) of the standard normal distribution; \( f(M^*) = \) original probability density ordinates at \( M^* \). The CDF and PDF of a Type I asymptotic variable \( M \) are functions of \( \alpha \) and \( u \) [(7)], and are given by

\[
F(M) = \exp(-e^{-\alpha(M-u)}); \quad f(M) = \alpha e^{-\alpha(M-u)} F(M) \tag{9a,b}
\]

The mean and standard deviation (StDev) values shown in Fig. 4 are equivalent normal values, calculated using (6), (8), and (9), which have been entered into the cells. For the equivalent mean and StDev of \( M \), the spreadsheet software Microsoft Excel (version 5 or 7) has the functions NORMSINV for the inverse of the CDF of standard normal, and NORMDIST for CDF or PDF of normal or standard normal. These functions have been used in connection with (8a) and (8b).

The \( x \)-values of \( Y, Z, \) and \( M \) were initially equal to the mean values 40, 50, and 1,000, respectively. Solver was then invoked, to Minimize \( \beta \). By Changing \( x \)-values, Subject To performance function \( g(x) = 0 \). The value of \( \beta \) was found to be 2.665 after optimization using the spreadsheet’s Solver tool.

The mathematical approach described in Ang and Tang (1984) involved orthogonal transformation to obtain a set of uncorrelated transformed variables, followed by partial differentiation of the transformed performance function and iteration in accordance with Rackwitz’s (1976) algorithm. An alternative method of obtaining the required independent set of equivalent (standard) normal variates using the Rosenblatt transformation was also described, yielding a \( \beta \) value of 2.663.

NOTES ON WORKING WITH SOLVER ADD-IN

An optimization tool resides in spreadsheet software like Microsoft Excel (where it is named Solver), Lotus 123, and Quattro Pro. This paper uses Microsoft Excel, the Windows 95 version of which has on-line help on Solver algorithm, options, completion messages, and other information.

The way Microsoft Excel’s Solver tool performs its analysis can be adjusted in the Solver Options dialog box. The default setting is: precision = 1e-6, tangent estimates, forward derivatives, and Newton search. Other available options include quadratic estimates, central derivatives, and conjugate search. These options are described in Solver Options Help file. The default setting is generally adequate, at least for the cases presented above. Also, when Solver reports a converged solution, the solution can often be improved by rerunning Solver (based on the converged solution) until it found a solution.

Steps 2–4 in Example 1 can also be condensed into a single step using a single array formula for \( \beta \), without displaying the intermediate matrix operations. This shortcut is particularly convenient when the number of random variables is large.

SUMMARY AND CONCLUSIONS

An efficient method using a spreadsheet software has been proposed for calculating the Hasofer-Lind second moment reliability index. The method is based on the perspective of an ellipsoid that is tangent to the failure surface in the original space of the variables. Correlation is accounted for by setting up the quadratic form of a tilted ellipsoid in the spreadsheet. Nonnormals are dealt with using established relationships between nonnormal distribution and its equivalent normal distribution. Iterative searching and numerical partial differentiation are performed automatically by a spreadsheet’s optimization tool. The implementation of the proposed method has been illustrated using two example problems. The results are virtually identical to those obtained using an established mathematical approach, except that the proposed method is simpler to implement. The image of an ellipsoid expanding in the original space of the variables also provides an intuitive grasp of the meaning of reliability index.

The proposed method, being relatively simple and intuitive,
may be an attractive alternative to the established mathematical procedures that require transformed space and close-form partial derivatives, especially for cases with either no explicit performance functions, or with lengthy and complicated performance function [e.g., Low (1997)].

APPENDIX. REFERENCES
