Basics of Set Theory

A set is understood to be a collection of certain well-defined objects, which in our view or conception composes a whole; these objects are described as elements of set A.

Written:
\( a \in A \) "a is an element of set A"
\( b \notin A \) "b is not an element of set A"

The definition of set A can occur in the following ways:

- explicitly, through the specification of all elements of the set, for example \( M = \{a, b, \ldots, z\}\); this is only possible for finite sets.

- implicitly, through the specification of a predicate, that is, a characteristic feature shared by all elements of the set. For example:
  \( M = \{x \mid x \in \mathbb{Z} \text{ and } x > 0; \mathbb{Z} = \text{set of natural numbers}\}\)
in this manner it is also possible to define infinite sets.

The number \( |M| \) of elements in a set M is called the cardinality of M.

Relationships between sets:

Equality:
\( A = B \iff \) every element A is also an element of B and vice versa;
also functions for \( A \neq B \)

Subset:
\( A \subseteq B \iff (x \in A \Rightarrow x \in B) \) for all \( x \in A \)
\( A \subset B \iff A \subseteq B \text{ and } A \neq B \)
(true subset)
Transitivity:
A ⊆ B and B ⊆ C ⇒ A ⊆ C

Special Sets:

Empty Set: \( \emptyset = \{ x | x \neq x \} \), also: \( \{ \} \)

Power Set: \( P(A) = \{ X | X \subseteq A \} \) Set of all subsets of A
Example: \( A = \{1,2\} \rightarrow P(A) = \emptyset, \{1\}, \{2\}, \{1,2\} \)

Product Set (Cartesian Product):
\( A \times B := \{(x,y) | x \in A \text{ and } y \in B \} \)
Set of all ordered tuples with ‘Coordinates’ \( x,y \)
General: \( A \times B \times C, \text{ etc.} \)
Specific: \( A \times A \times A \ldots \times A := A^n \)
for example: \( \mathbb{R}^3 \) three-dimensional, real number space (vector space)

Algebraic Operations of Sets:
Let \( A, B, C, \ldots \in P(M) \) be elements of set \( M \)

Intersection: \( A \cap B = \{ x | x \in A \text{ and } x \in B \} \)
A and \( B \) are disjoint, if \( A \cap B = \emptyset \)

Union: \( A \cup B = \{ x | x \in A \text{ or } B \} \)

Relative Complement: \( A \setminus B = \{ x | x \in A \text{ and } x \notin B \} \)
"A without B"
Some Laws of Set Algebra:

Commutativity: \( A \cap B = B \cap A \) (is equally valid for \( \cup \) )

Associativity: 
\( (A \cap B) \cap C = A \cap (B \cap C) \)
(is equally valid for \( \cup \) )

Distributivity: 
\( A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \)
\( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \)

Relations:
A subset \( R \subseteq X \times Y \) of the cartesian product \( X \times Y \) is a two-digit (binary) relation \( R \) between sets \( X \) and \( Y \).

The relation \( (x,y) \in R \) can also be written as: \( xRy \)

N-ary relations are defined analogously.

Example 1:
"\(<" is a binary relation in \( \mathbb{R}^2 \): \( (x,y) \in "<" \) or rather, \( x < y \)

Example 2:
"lies between" = \{ \( (a,b,c) \mid a, b, c \in G \) and \( c \) is a point on the line \( g(a,b) \) \}
\( \subseteq G \times G \times G \) for the points in \( G \) a two-dimensional area \( G \)

Characteristics of Relations:

Reflexive: \( xRx \) e.g. "\( \leq \)" is reflexive

Symmetric: \( aRb \leftrightarrow bRa \) e.g. "\( = \)" is symmetric

Antisymmetric: \( aRb \) and \( bRa \) \( \rightarrow \) \( a = b \) e.g. "\( \leq \)" is antisymmetric

Transitive: \( aRb \) and \( bRc \) \( \rightarrow \) \( aRc \) e.g. "\( < \)" is transitive

\( R \) is an equivalence relation if \( R \) is reflexive, symmetric and transitive. The set of elements which are equivalence relations compose an equivalence class: \( R[x] = \{ y \mid (x,y) \in R \} \). Therefore, one can also form partitions (class separations) in a set.
Example:
"parallel" is an equivalence relation in the set of all lines in one plane; as a result, classes of parallel lines can be formed.

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R is an ordered relation if R is reflexive, antisymmetric and transitive (e.g. ≤). Ordered relations are important for the sorting of data by size.