First-Order Reliability Methods (FORM)
-First Order Second Moment Method

The performance function of a system can be written as

\[
g(X_1, X_2, \cdots, X_n) \begin{cases} 
> 0 & \text{safe state} \\
= 0 & \text{limit state} \\
< 0 & \text{failure state}
\end{cases}
\]

where \( g(X) = 0 \) is known as a limit state surface and each \( X \) indicates the basic load or resistance variable. Usually, a number of limit states can be identified for a system, with each representing a state of either ultimate system failure, system unserviceability, or operational malfunction. \( X \) such that \( g(X) = 0 \) is the level set of \( g(\cdot) \) at level 0.

For ease of analytical development, all the variables are transformed into their standardized forms.

\[
X'_i = \frac{X_i - \bar{X}_i}{\sigma_i}
\]

Obviously, the expected value and standard deviation of \( X'_i \) are, respectively, zero and unity. The limit state equation must then correspondingly be written in terms of the standardized variables as follows;

\[
g(X'_1, X'_2, \cdots, X'_n) \begin{cases} 
> 0 & \text{safe state} \\
= 0 & \text{limit state} \\
< 0 & \text{failure state}
\end{cases}
\]

In general, \( g(X') \) is a nonlinear function. However, it can be linearized at some point \( X' \).
\[ g(X') = g(x'^0) + \nabla g(x'^0) \cdot (X' - x'^0) + o\|X' - x'^0\| \]

where

\[ \nabla g(x'^0) = \left[ \frac{\partial g}{\partial x_1}(x'^0) \ldots \frac{\partial g}{\partial x_n}(x'^0) \right]^T \]

Neglecting higher order terms and assuming that X's are statistically uncorrelated, the expected value and standard deviation of \( g(X') \) is given by

\[ \bar{g}_0 = g(x'^0) - \nabla g(x'^0) \cdot x'^0 \]

\[ \sigma_{\bar{g}_0}^2 = \nabla g(x'^0)^T \nabla g(x'^0) \]

\[ \beta_0 = \frac{\bar{g}_0}{\sigma_{\bar{g}_0}} \]

If X's are correlated, we can use the principal component transformation before standardization.

**Mean Value First Order Second Moment (MVFOSM)**

The MVFOSM method derives its name from the fact that it is based on a first-order Taylor series approximation of the performance function linearized at the mean values of the random variables, and because it uses only second-moment statistics of the random variables. In such a case, \( x'^0 = 0 \), and

\[ \bar{g} = g(x' = 0) \]

\[ \sigma_g^2 = \nabla g(x' = 0)^T \nabla g(x' = 0) \]

\[ \beta = \frac{\bar{g}}{\sigma_g} \]
Deficiency of MVFOSM

The function $g(-)$ is linearized at the mean values of $X_i$ variables. When $g(-)$ is nonlinear, significant error may be introduced by neglecting higher order terms. More importantly, the safety index fails to be constant under different but mechanically equivalent formulations of the same performance function. Such an invariance problem is circumvented if the first-order approximations are evaluated at a point on the failure surface.

Example (Example 7.3 in Haldar & Mahadevan, 2000)

A W16X31 steel section made of A36 steel is suggested to carry an applied deterministic bending moment of 1140 kip-in.

$\mu_{F_y} = 38$ ksi, $\sigma_{F_y} = 3.8$ ksi

$\mu_Z = 54$ in$^3$, $\sigma_Z = 2.7$ in$^3$

It is quite logical to assume that $F_y$ and $Z$ are statistically independent

Strength Formulation

$g(-) = F_y Z - 1140 = (\mu_{F_y} + F_y' \sigma_{F_y}) \left(\mu_Z + Z' \sigma_Z\right) - 1140$

Using the MVFOSM method, $\beta = 3.975$

Stress Formulation

$g(-) = F_y - \frac{1140}{Z} = \left(\mu_{F_y} + F_y' \sigma_{F_y}\right) - \frac{1140}{(\mu_Z + Z' \sigma_Z)}$

Using MVFOSM method, $\beta = 4.282$

The limit state and the linearized limit state of the performance function are shown in Figure 1. Note that different equivalent formulations of performance function will
not change the failure (limit state) surface because the "equivalency" is based on \( g(-) = 0 \). However, linearized limit state depends on what formulation of performance function is used in mean value Taylor series expansion. This is because the mean values are not on the failure surface and two performance functions are different away from failure surface. Such an invariance problem is circumvented if the first-order approximations are evaluated at a point on the failure surface.

![Graph](image-url)

**Figure 1.** Limit State and Linearized Limit State of the Performance Function.

**Advanced First Order Second Moment Method (AFOSM)**

The invariance problem is circumvented if the first-order approximations are evaluated at a point on the failure surface. However, there are infinitely many points on the failure surface. The value of \( \beta_0 \) obviously depends on the point \( \mathbf{x}^0 \), around which the performance function is expanded. The minimum value \( \beta \) of \( \beta_0 \) is referred to as the reliability index. A corresponding point \( \mathbf{x}^0 \) is denoted by \( \mathbf{x}^* \) and designated either the design point or the checking point.
It can be shown that the reliability index defined by the linear performance function with uncorrelated random variables equals to the distance of the failure surface to the origin of the reduced (or standardized) variable. For non-linear performance function with uncorrelated random variables, the reliability index equals to the distance of the linearized failure surface to the origin of the reduced (or standardized) variable, as shown in Figure 2.

Figure 2. Determination of Design Point when Components of X are Uncorrelated
The design point, $x^{*}$ can be found by the following constrained optimization problem:

Minimize:  \[ \beta_0 = \left( x^{*T} x^{*} \right)^{1/2} \]

Subjected to the constraint:  \[ g(x^{*}) = 0 \]

The reliability index becomes

\[ \bar{g} = -\nabla g(x^{*})^{T} x^{*} \]

\[ \sigma_{\bar{g}}^2 = \nabla g(x^{*})^{T} \nabla g(x^{*}) \]

\[ \beta_{\text{Hill}} = \frac{\bar{g}}{\sigma_{\bar{g}}} = \frac{-\nabla g(x^{*})^{T} x^{*}}{\sqrt{\nabla g(x^{*})^{T} \nabla g(x^{*})}} \]

The reliability index such defined is called Hasofer-Lind reliability index for the credit of their first invention. Using the above $\beta_{\text{Hill}}$, the design point can be written as

\[ x^{*} = \frac{-\nabla g(x^{*}) \beta_{\text{Hill}}}{\sqrt{\nabla g(x^{*})^{T} \nabla g(x^{*})}} = -\alpha \beta_{\text{Hill}} \]

Where $\alpha$ is the direction cosine along the axes $x_{i}^{*}$

\[ \alpha = \frac{\nabla g(x^{*})^{T}}{\sqrt{\nabla g(x^{*})^{T} \nabla g(x^{*})}} \]

An algorithm was formulated by Rackwitz (1976) to compute $\beta_{\text{Hill}}$ and $x_{i}^{*}$ as follows:

1. Define the appropriate limit state equation
2. Assume initial values of $x_{i}^{*}$, $i = 1, 2, \ldots, n$ and obtain the reduced variates

\[ x_{i}^{*} = (x_{i}^{*} - \mu_{x_{i}}) / \sigma_{x_{i}} \]  Typically, the initial design point may be assumed to be at the mean values of the random variates.

3. Evaluate $\nabla g(x^{*})$ and $\alpha$ at $x^{*}$.
4. Express the new design point, $x^{*}$, in terms of $\beta_{\text{Hill}}$, \[ x^{*} = -\alpha \beta_{\text{Hill}}. \]
5. Substitute the new $x^*$ in the limit state equation $g(x^*)$, and solve for $\beta_{\text{ill}}$.
6. Using the $\beta_{\text{ill}}$ value obtained in Step 5, reevaluate $x^* = -\alpha \beta_{\text{ill}}$.
7. Repeat Step 3 through 6 until $\beta_{\text{ill}}$ converges.

This algorithm is shown geometrically in Figure 3. The algorithm constructs a linear approximation to the limit stat at every search point and finds the distance from the origin to the limit state. In Figure 3, Point B represents the initial design pint, usually assumed to be at the mean values of the random variables, as noted in Step 2. Note that B is not on the limit state equation $g(X^*) = 0$. The tangent to the limit state at B is represented by the line BC. Then AD will give an estimate of $\beta_{\text{ill}}$ in the first iterations, as noted in Step 5. As the iteration continues, $\beta_{\text{ill}}$ converges.

![Figure 3 Algorithm for Finding $\beta_{\text{ill}}$.](image-url)
If the components of $X'$ are correlated and have a covariance (correlation) matrix, $\Sigma$ (symmetric), a standardized random vector, $Z$, whose components are uncorrelated can be constructed by principal component transformation

$$Z = \Phi X'$$

where $\Phi \Sigma \Phi^T = I$ or $\Sigma = \Phi^T \Phi$

In this case, the limit state equation must also be transformed for the estimation of the limit state probabilities into

$$g(\Phi^T Z) = g_i(Z) \leq 0$$

Then, the random vector, $Z$, with uncorrelated components and the limit state equation in terms of $Z$, form exactly the same analytical problem as solved for $X'$ and $g(X') \leq 0$.

It can be shown that

$$\beta_{HL} = \sqrt{z^* z^*} = \sqrt{x^* x^*} \Sigma^{-1}$$

Hence, the safety index for the cases involving $x$ with correlated components can also be derived without transforming $X'$ into $Z$. The design point, $x^*$ can be found by the following constrained optimization problem (as shown in Figure 4):

Minimize: $\beta_0 = (x^* x^*)^{1/2}$

Subjected to the constraint: $g(x^*) = 0$

If a random vector $X'$ is Gaussian with uncorrelated (independent) components, the joint pdf is given by
while if $X'$ is Gaussian with correlated components

$$f(x') = \frac{1}{\sqrt{2\pi}^n|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}x'^\top \Sigma^{-1}x'\right)$$

It follows from the definition of the reliability index that the design point $x'^* \in X$ is the point of maximum likelihood (Figure 2 and Figure 4); and the reliability index represents the shortest distance between the origin and the limit state surface in $Z$ space.

![Diagram](image_url)

Figure 4 Determination of Design Point when Components of X are Correlated
Estimation of Limit State Probabilities

Figure 5 Tangent plane to $g(X)=0$ at $x^{**}$
Limit State under Multiple Limit Conditions

Figure 6 Limit State under Multiple Limit Conditions